Recursive Lagrangian Dynamics of Flexible Manipulator Arms via Transformation Matrices

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Much work has been done to formulate the dynamic equations of motion for mechanical arms with rigid links. Work on the "inverse dynamic formulation" used in control can be found in references [22], [27], [29], [2] and in their bibliographies. References [30], [20], [33], [32], [12], and their bibliographies represent work on the dynamic formulation for simulating rigid link arms. The efficiency of these formulations and alternatives to their real time calculation is discussed in [26], [1] and the works referenced therein.

The limitation of these works is that rigid links are assumed. With this assumption the techniques become at some point self defeating, if their purpose is to improve performance. Maintaining rigidity of the links inhibits improved performance but is necessary if the rigid link assumption is to be accurate.

Consideration of flexibility and control of the links in arm-type devices was reported in 1972 by Mirro [24]. This early work considered both the modeling and control of a single link device. Book [7] considered the linear dynamics of spatial flexible arms represented as lumped mass and spring components via 4x4 transformation matrices. This was refined and later reported in [9]. Book and Whitney [3], [4] later considered linear distributed dynamics of planar arms via transfer matrices and the limitations flexibility imposed on control system performance [8]. Maizza and Whitney [23], [4] used a planar nonlinear model with modal representation of the flexibility and considered modal control as a technique for overcoming the limitations of the flexibility. Whitney, Book, and Lynch [34], [4] considered the design implications of flexibility. Distributed frequency domain analysis of nonplanar arms using transfer matrix techniques [5], [6] has been used by Book, et.al to verify the accuracy of truncated modal models of the nonlinear spatial dynamics of flexible manipulators (the Remote Manipulator of the Space Shuttle). The nonlinear modal model appearing here was first presented by the author in 1982 [10]. A more classical approach to manipulator dynamics, both rigid [18] and flexible [19], has been undertaken by Huston and his coworkers.

The work in flexible spacecraft has spawned a line of research pertaining to the interaction of articulated structures. This work has great relevance to the manipulator modeling problem. Entries into this literature are provided by the works of Likins [21] and Hughes [25]. This activity produced a spatial, nonlinear, flexible manipulator model reported by Ho et.al. [14] and corresponding computer code for simulation. The simulation required great amounts of computer time and was unsuitable for even off line simulation. Further work for the purposes of simulating the Space Shuttle Remote Manipulator was performed by Hughes. His linearized model is reported in [16] and a more general model is reported in [17]. The Hughes model ignores the interaction between structural deformation and angular rate as might be appropriate for the Space Shuttle arm. This work and associated work at SPAR Aerospace, Ltd. and the Charles Stark Draper Laboratory, Inc. probably represent the most intensive work on the modeling, simulation and control of flexible arms. Unfortunately, little of this work has been reported in the open literature. Recent examination of experimental results from the operation of the Shuttle arm in space has confirmed the validity of these models. More recently, Singh and Likins [28] have reported an efficient flexible arm simulation program.

Yet another branch of research that has found its way to the flexible manipulator dynamics problem is the study of flexible mechanisms. Dubowsky and Gardner [13] and Winfrey [35] provide the reader with a bibliography on this work. Sunada and Dubowsky [31] have developed modeling techniques applicable to both spatial closed loop mechanisms and open loop chains such as manipulator arms. This work assumes a known nominal motion over time about which the flexible arm equations are linearized. This falls short of a true simulation of the flexible, nonlinear equations, but is an interesting compromise for the sake of computational speed. This technique is oriented toward finite element analysis to obtain modal characteristics of the links which are then combined using a time varying compatibility matrix. It uses 4x4 matrices to represent the nominal kinematics and derivation of the compatibility matrix.

1.1. Perspective on This Work

This report stresses an efficient, complete, and conceptually straightforward modeling approach using the 4x4 transformation matrices that are familiar to workers in the field of robotics. It is unique in several respects: It uses 4x4 matrices to represent both the joint and deflection motion. The deflection transformation is represented in terms of a summation of modal shapes. The computations resulting from the I.agrangian formulation of the dynamics arc reduced to recursive form similar to that which has proven so efficient in the rigid link case. The equations are free from assumptions of a nominal motion, and do not ignore the interaction of angular rates and deflections. They do assume small deflections of the links which can be described by a summation of the modal shapes and a linear model of elasticity. Only rotational joints arc allowed. The results are quite tractable for automated computer solution of arbitrary rotary joints. Preliminary programs written to evaluate computational efficiency show that this method requires about 2.7 times as many computations as the most efficient rigid formulations with the same number of degrees of freedom. The rigid model could incorporate 21 degrees of freedom compared to 12 degrees of freedom (6 of which arc joints) for this flexible model. Thus, 15 degrees of freedom in the rigid model could be used to approximate the flexibility that the 6 flexible degrees of freedom of the model presented here approximate. The relative accuracy of the two approximations has not been determined. These issues are discussed in more detail in the Conclusions.

2. Flexible Arm Kinematics

The previous works on rigid arm dynamics use the serial nature of manipulator arms which results in multiplicative terms in the kinematics. The modal representation of flexible structure dynamics, on the other hand, is a parallel or additive representation of the system behavior. One of the contributions of this paper is to resolve this difference in a concise way. As with many of the previous works on rigid dynamics, die 4x4 matrices of Denavit and Hartenberg [11] arc used. Sunada and Dubowsky [31] used this representation for their flexible arm simulations but did not produce a complete nonlinear dynamic simulation. Other workers such as Hughes [17] relied on the more general formulation provided by a vector-dyadic representation. While Silver [27], Hollerbach [15], and others have pointed out the relative inefficiency of the 4x4 formulation, the conceptual framework is most advantageous when tackling the complexity of the flexible dynamics.

Define the position of a point in Cartesian coordinates by an augmented vector:

[1 x-component y-componcnt /-component]^{T}.

Define the coordinate system $[x\ y\ z]_i$ on link i with origin O_i at the proximal end (nearest the base) oriented so that the x axis is coincident with the neutral axis of the beam in its undeformed condition. The orientation of the remaining axes will be done so as to allow efficient description of the joint motion. A point on the neutral axis at x = ?] when the beam is undeformed is located at 'h^ij) under a general condition of deformation, in terms of system i.

By a homogeneous transformation of coordinates the position of a point can be described in any other coordinate system j if the transformation matrix $^{j}W_i$ is known. The form of this matrix is

$$ {}^{j}W_{i} = \begin{bmatrix} 1 & | & 0^{T} \\ x_{j} \text{ component of } O_{i} & | & \\ y_{j} \text{ component of } CX & | & JR_{l} \\ z_{j} \text{ component of } O_{l} & | & \end{bmatrix} \quad (1) $$

where

${}^{j}R_{i}$ = a 3x3 matrix of direction cosines
0 = a 1x3 vector of zeros.

Thus in terms of the fixed incrtial coordinates of the base the position of a point on link i is given as

$$ h_{i}^{!} = {}^{o}\backslash V_{i}^{1} {}^{j}h_{i}^{1} = W_{i}^{1} {}^{j}h_{i}^{1} \quad \mathbf{(2)} $$

where die special case of ${}^{0}W_{j} = W_{i}$. It is useful to separate the transformations due to the joint from the transformation due to the flexible link as follows

$$ W_{j} = W_{j-1} F_{j-1} A_{j} = \hat{W}_{j-1} A_{j} \quad (2) $$

where

A_{j} = the joint transformation matrix for joint j
K_{j-1} = the link transformation matrix for link j-1 between joints j-1 andj
W_{j-1} = the cumulative transformation from base coordinates to O_{j-1} at the distal end of link j.

6. , is fixed to the link j-1 and with no deflection $[\hat{x}\ \hat{y}\ \hat{z}]_{j-1}$ is parallel to $[x\ y\ z]_{j-1}$, with x_{j-1} coincident with $\hat{x}_{j-1}$.

To incorporate the deflection of the link, the approach of modal analysis is used which is valid for small deflection of the link.

$$ {}^{i}h_{i}(\eta) = \begin{bmatrix} 1 \\ n \\ 0 \\ 0 \end{bmatrix} + \sum_{j=1}^{m_{j}} \delta_{ij} \begin{bmatrix} 0 \\ x_{ij}(\eta) \\ y_{ij}(\eta) \\ z_{ij}(\eta) \end{bmatrix} \quad \mathbf{(4)} $$

where

x_{ij}, y_{ij}, z_{r} = the x_{r} y_{i}, and z_{j} displacement components of mode j of link Fs deflection, respectively.
5_{ij} = the time varying amplitude of mode j of link i
m_{i} = the number of modes used to describe the deflection of link i.

The link transformation matrix must also incorporate the deflection of the link. Here the rotations as well as the translations of die deflection must be represented. If one consistently requires small rotations the direction cosine matrix simplifies as noted in [9] and furthermore the small angles can be assumed to add vectorally. This is basic to the approach used here. The link transformation matrix can then be written as

$$ F_{i} = \left[H_{i} + \sum_{j=1}^{m_{i}} \delta_{ij} M_{ij} \right] \quad \mathbf{(5)} $$

$$\mathbf{H}_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_i & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

$$\mathbf{M}_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ x_{ij} & 0 & -\theta_{zij} & \theta_{yij} \\ y_{ij} & \theta_{zij} & 0 & -\theta_{xij} \\ z_{ij} & -\theta_{yij} & \theta_{xij} & 0 \end{bmatrix} \tag{7}$$

and where

All variables in brackets are evaluated at l_i

$\theta_{xij}, \theta_{yij}, \theta_{zij}$ = the x_i, y_i, and z_i rotation components of link i, respectively.

l_i = the length of link i.

To find the velocity of a point on link i, take the time derivative of the position:

$$\frac{d}{dt}\mathbf{h}_i = \dot{\mathbf{h}}_i = \dot{\mathbf{W}}_i\, {}^i\mathbf{h}_i + \mathbf{W}_i\, {}^i\dot{\mathbf{h}}_i. \tag{8}$$

Due to the serial nature of the kinematic chain, it is computationally efficient to relate the position of a point and its derivatives to preceeding members in the chain. By differentiating 2 one obtains:

$$\dot{\mathbf{W}}_j = \dot{\hat{\mathbf{W}}}_{j-1}\, \Lambda_j + \hat{\mathbf{W}}_{j-1}\, \dot{\Lambda}_j \tag{9}$$

$$\ddot{\mathbf{W}}_j = \ddot{\hat{\mathbf{W}}}_{j-1}\, \Lambda_j + 2\, \dot{\hat{\mathbf{W}}}_{j-1}\dot{\Lambda}_j + \hat{\mathbf{W}}_{j-1}\, \ddot{\Lambda}_j \tag{10}$$

where

$$\dot{\Lambda}_j = \mathbf{U}_j\, \dot{q}_j \tag{11}$$

$$\ddot{\Lambda}_j = \mathbf{U}_{2j}\dot{q}_j^2 + \mathbf{U}_j\ddot{q}_j \tag{12}$$

$$\mathbf{U}_j = \partial\Lambda_j/\partial q_j$$

$$\mathbf{U}_{2j} = \partial^2\Lambda_j/\partial q_j^2$$

q_j = the joint variable of joint j.

Thus $\dot{\mathbf{W}}_j$ and $\ddot{\mathbf{W}}_j$ can be computed recursively from $\hat{\mathbf{W}}_{j-1}$, its derivatives, and the partials with respect to the variables of link j-1 and joint j. No mixed partials are explicitly present. This computational approach is similar to that proposed by Hollerbach [15] for rigid link arms. Here one additionally needs $\hat{\mathbf{W}}_{j-1}$ and its derivatives. These can be computed recursively from $\mathbf{W}_{j-1}$ and its derivatives:

$$\hat{\mathbf{W}}_j = \mathbf{W}_j\, \mathbf{E}_j \tag{13}$$

$$W_j = W_j K_j + 2 W_j ^ + W_j K_j \qquad (15)$$

$$\dot{K}_j = \sum_{k=1}^{m_j} \dot{\delta}_{jk} M_{jk} \qquad \textbf{(16)}$$

$$\ddot{K}_j = \sum_{k=1}^{m_j} \ddot{\delta}_{jk} M_{jk} \qquad (17)$$

The last two equations illustrate how the deflection transformations enter even more simply into the kinematics on a per variable basis than do the joint variables. This is due to the small deflection assumption and the form chosen for the transformation. The recursive nature of the velocity and acceleration is preserved from the rigid case. For the simulation equations the terms involving second derivatives of the joint and deflection variables will be separated from the above expressions and included in the inertia matrix to make up the coefficient matrix of the derivatives of the state variables. The "inverse dynamics" solution that proceeds directly from the Lagrangc formulation has little obvious utility.

3. System Kinetic Energy

In this section the expression for the system kinetic energy is developed for use in Lagrangc's equations. First, the kinetic energy for a differential element is written. Then, integration of this differential kinetic energy over die link gives the link's total contribution. This produces terms that arc the equivalent of the moment of inertia matrices of rigid link arms. Summation over all the links provides the total kinetic energy.

'lTic kinetic energy of a point on the i-th link is

$$dk_i = \frac{1}{2} dm \, Tr\{ \dot{h}_i \dot{h}_i^T \} \qquad (18)$$

where

dm is the differential mass of the point and
Tr{.} is the trace operator.

Hxpanding 18 and using the fact that $Tr\{A B^T\} = Tr\{B A^T\}$> the expression for dk_i becomes

$$dk_i = \frac{1}{2} dm \, Tr\{ \dot{W}_i \, {}^ih_i \, V_i \, V/J + 2 \dot{W}_i^1!]_i \, {}^{[}ti_i*' \, V/J + W_i \, \backslash \, {}^j\dot{h}_i^r \, \backslash\backslash J \} \qquad (19)$$

where

$$ {}^i\dot{h}_i = \sum_{j=1} \dot{\delta}_{ij} [\, 0 \;\; x_{ij} \;\; y_{ij} \;\; z_{ij} \,]^T. \qquad \textbf{(20)}$$

By integrating over the link one can obtain the total link kinetic energy. In this report it is assumed that the links arc slender beams because it makes the central development clearer. Other mass distributions could be used with a slight departure here in the development. For slender beams dm = /x di] and one can integrate over $t)$ from 0 to L. Only the terms in ${}^ih_i$ and its derivatives arc functions of $i)$ for this link. Thus the integration can be performed without knowledge of W_i and its derivative. Summing over all n links one finds the system kinetic energy to be

$$K = \sum_{i=1}^{n} \int_0^{l_i} dk_i \tag{21}$$

$$K = \sum_{i=1}^{n} \mathrm{Tr}\left\{ \dot{W}_i B_{3i} \dot{W}_i^T + 2\dot{W}_i B_{2i} W_i^T + W_i B_{1i} W_i^T \right\} \tag{22}$$

where

$$B_{1i} = \frac{1}{2}\int_0^{l_i} \mu \, {}^i\dot{h}_i \, {}^i\dot{h}_i^T \, d\eta. \tag{23}$$

By interchanging the integration in 23 and the summations involved in the definition of ${}^i\dot{h}_i$ in 20 one obtains

$$B_{1i} = \sum_{j=1}^{m_i} \sum_{k=1}^{m_i} \dot{\delta}_{ij} \dot{\delta}_{ik} C_{ikj} \tag{24}$$

where

$$C_{ikj} = \frac{1}{2}\int_0^{l_i} \mu \, [0 \; x_{ik} \; y_{ik} \; z_{ik}]^T [0 \; x_{ij} \; y_{ij} \; z_{ij}] \, d\eta. \tag{25}$$

C_{ikj} has units of an inertia matrix and serves a similar function. While shown here as a 4x4 matrix it is nonzero only in the 3x3 (lower right). It can also be shown that $C_{ikj} = C_{ijk}^T$. By choosing the assumed mode shapes in an appropriate manner, it is possible to reduce the number of nonzero terms in 24. This matter is discussed in light of computational speed in the conclusions.

The other terms in equation 22 can similarly be found:

$$B_{2i} = \frac{1}{2}\int_0^{l_i} \mu \, {}^i h_i \, {}^i\dot{h}_i^T \, d\eta \tag{26}$$

$$B_{2i} = \sum_{j=1}^{m_i} \dot{\delta}_{ij} C_{ij} + \sum_{k=1}^{m_i} \sum_{j=1}^{m_i} \delta_{ik} \dot{\delta}_{ij} C_{ikj} \tag{27}$$

where

$$C_{ij} = \frac{1}{2}\int_0^{l_i} \mu \, [1 \; \eta \; 0 \; 0]^T [0 \; x_{ij} \; y_{ij} \; z_{ij}] \, d\eta. \tag{28}$$

Finally, by a similar approach:

$$B_{3i} = \frac{1}{2}\int_0^{l_i} \mu \, {}^i h_i \, {}^i h_i^T \, d\eta$$

$$B_{3i} = C_i + \sum_{j=1}^{m_i} \delta_{ij} [C_{ik} + C_{ik}^T] + \sum_{k=1}^{m_i} \sum_{j=1}^{m_i} \delta_{ik} \delta_{ij} C_{ikj} \tag{29}$$

$$C_i = \pm J \quad /illijOOJ'llijOOJdij. \tag{3U}$$
2 0

This final term contains the rigid body inertia terms.

It should be noted that these terms are easily simplified if one link in the system is to be considered rigid, in which $m_i = 0$. Should a link consist of a flexible member with rigid appendages the above derivation is readily extended to modify the matrices C_{ikj}, C_{ik}, and C_i with no further modifications to the succeeding development. In fact, these matrices could be obtained by finite clement analysis should the link shape be irregular as is often the case. Furthermore, the expression for B_r contains a term of order 5^2 which is by definition small and a candidate for later elimination. Finally, much of the complexity of the integration of the modal shape products can be done offline, once, for a given link structure,

3.1. Derivatives of Kinetic Energy

For construction of I .agrange's equations one needs

$$3K/3q_j,\quad 3K/3\delta_{jf},\quad \frac{d}{dt}(3K/3\dot{q}_j),\quad \text{and}\ \frac{d}{dt}(3K/3\dot{\delta}_{jf})$$

First consider $3K/3\dot{q}_j$. This will involve the partials of all the terms in 22, some of which arc zero. In fact, only $\dot{W}_i$ for $j \leq i \leq n$ provides nonzero partials with respect to $\dot{q}_j$. The time derivative of die partial is then taken. In this respect the following equivalences should be noted:

$$d\dot{X}Y./d\dot{q}\wedge \quad 3W_i/3q_J \tag{31}$$

$$\frac{d}{dt}(a\dot{w}_i/a\dot{q}_j) \wedge a\dot{w}_i/aq_j \tag{32}$$

$$3\dot{W}_i/3\dot{\delta}_{jf} = aw_r/3\delta_{jf} \tag{33}$$

$$\frac{d}{dt}(3\dot{W}_i/3\dot{\delta}_{jf}) = 3\dot{W}_i/3\delta_{jf}. \tag{34}$$

Also helpful in simplifying the result is that, $Tr\{A\} = Tr\{A^T\}$ for any square matrix A and that B_{3i} is symmetric. Considerable cancellation and combination results when the terms in Lagrange's equation involving the kinetic energy are combined. The result of this combination is

$$\frac{d}{dt}(3K/3\dot{q}_j) - 3K/3q_j =$$

$$2\sum_{i=j}^{n} Tr\{\frac{\partial}{\partial q_j} - [[C_i + \sum_{k=1}^{m_i} \delta_{ik}(C_{ik} + C_{ik}^T + \sum_{l=1}^{m_i} \delta_{il} C_{ilk})]\ddot{W}_i^T$$

$$+ [\sum_{k=1}^{m_i} \quad \sum_{l=1}^{m_j} \quad \sum_{k=1}^{m_i} \quad \sum_{l=1}^{m_j} \quad]\dot{W}_i^T]\} \tag{35}$$

assumption that the deflections are small. Noting the recurrance of certain terms above, it is convenient to define the following:

$$D_{ik} = C_{ik} + \sum_{l=1}^{m_i} \delta_{il} C_{ilk} \tag{36}$$

$$G_i = C_i + \sum_{k=1}^{m_i} \delta_{ik} (C_{ik} + C_{ik}^T). \tag{37}$$

When these definitions are substituted into equation 35 one obtains:

$$\frac{d}{dt} (\partial K / \partial \dot{q}_j) - \partial K / \partial q_j =$$

$$2 \sum_{i=j}^{n} Tr \{ \frac{\partial W_i}{\partial q_j} [G_i \ddot{W}_i^T + \sum_{k=1}^{m_i} \ddot{\delta}_{ik} D_{ik} W_i^T + 2 \sum_{k=1}^{m_i} \dot{\delta}_{ik} D_{ik} \dot{W}_i^T] \} . \tag{38}$$

The partials of K, with respect to δ_{jf} and $\dot{\delta}_{jf}$, are considerably more complex due to the fact that B_{1i}, B_{2i}, and B_{3i} are functions of the deflection variables. The techniques of simplification are similar. An additional simplification arises due to the fact that if A were any antisymmetric matrix, and if W were a matrix compatible for multiplication, then $Tr\{ W A W^T \} = 0$. An antisymmetric matrix occurs from the difference of a matrix and its transpose.

$$\frac{d}{dt} (\partial K / \partial \dot{\delta}_{jf}) - \partial K / \partial \delta_{jf} =$$

$$2 \sum_{i=j+1}^{n} Tr \{ \frac{\partial W_i}{\partial \delta_{jf}} [G_i \ddot{W}_i^T + \sum_{k=1}^{m_i} \ddot{\delta}_{ik} D_{ik} W_i^T + 2 \sum_{k=1}^{m_i} \dot{\delta}_{ik} \dot{D}_{ik} \dot{W}_i^T \} +$$

$$Tr \{ 2 [\ddot{W}_j D_{jk} + 2 \dot{W}_j \sum_{k=1}^{m_j} \dot{\delta}_{jk} C_{jkf} + W_j \sum_{k=1}^{m_j} \ddot{\delta}_{jk} C_{jkf}] W_j^T \}. \tag{39}$$

4. System Potential Energy

The potential energy of the system arises from two sources: elastic deformation and gravity. In both cases they are included by first writing the potential energy contribution of a differential element, integrating over the length of the link, and then summing over all links.

4.1. Elastic Potential Energy

Consider a point on the i-th link undergoing small deflections. First restrict the link of the slender beam type. The elastic potential is accounted for to a good approximation by bending about the transverse y_i and z_i axes and twisting about the longitudinal x_i axis. Compression is not initially included since it is generally much smaller. Along an incremental length $d\eta$ the elastic potential is

$$dv_{ci} = \pm d,\{ K [.,(\wedge Y + I_y(- \wedge)^2] + G I_x (*?*.?\} \quad (40)$$

where

θ_{xi}, $\#_y$,j, and $\#_{zi}$ arc the rotations of the neutral axis of the beam at the point i) in the x_f, y_v and z_1 directions, respectively. Since deflections arc small, these directions arc essentially parallel or perpendicular to the neutral axis of the beam.

H = Young's modulus of elasticity of the material

G = The shear modulus of the material

I_x = The polar area moment of inertia of the link cross section about the neutral axis.

I_y, I_z = the area moment of inertia of the link cross section about the y_1 and z_1 axes, respectively.

With a truncated modal approximation for the beam deformation the angles $\#_{xj}$, $\$_{yi}$, and $6_{/x}$ are represented as summations of modal coefficients times the deflection variables. The x rotation, for example is

$$\theta_{xi} = \sum_{k=1}^{m_i} \delta_{ik}\theta_{xik} . \quad (41)$$

where $0\wedge_k$ is the angle about the $x_{\{}$ axis corresponding to the k-th mode of link i at the point *it).* When dv_{ci} is integrated over the link the integration can be taken inside the modal summations of equation 41 and its corresponding y and z components. The following definitions then prove useful:

$$K_{ikl} = K_{xikl} + K_{yikl} + K_{zikl} , \quad (42)$$

where

$$K_{xikl} = \int_0^{l.} \frac{\quad_{xil}}{\partial\eta} \frac{\partial\theta_{xik}}{\partial\eta} d\eta \quad (43)$$

$$K_{yikl} = \int_0^{l_i} E I_y(\eta) \frac{\partial\theta}{\partial} .30 .t$$

$$K_{zikl} = \int_0^{>i} H 1_7 (TJ) \frac{\wedge\wedge}{\partial\eta} \& \frac{\wedge a}{\partial\eta} \wedge drj \quad (45)$$

Note that K_{jkl} = K_{jlk} and that for certain special cases the orthorgonality of the modal functions can eliminate many of the terms in equations 43, 44, and 45. The clastic potential for the total system, V_c can then ***ho* writton :is**

Note that the V_e is independent of q_i, the joint variables.

$$\frac{\partial V_e}{\partial q_j} = 0. \tag{47}$$

For deflection variables

$$\frac{\partial V_e}{\partial \delta_{jf}} = \sum_{k=1}^{m_j} \delta_{jk} K_{jkf} . \tag{48}$$

The form of equation 48 is much more general than the initial assumptions made regarding the contributions to the elastic potential energy would allow. Compression strain energy, and link forms other than beams can be represented in this form. The values of the coefficients K_{jkf} can be determined analytically or numerically, eg. by finite element methods.

4.2. Gravity Potential Energy

For a differential element on the i-th link of length $d\eta$ the gravity potential is

$$dv_{gi} = -\mu\, g^T W_i \, {}^i h_i \, d\eta , \tag{49}$$

where the gravity vector g has the form

$$g^T = [\, 0 \;\; g_x \;\; g_y \;\; g_z \,] .$$

When integrated over the length of the beam and summed over all beams, the gravity potential becomes

$$V_g = - g^T \sum_{i=1}^{n} W_i \, r_i \tag{50}$$

where

$$r_i = M_i \, r_{ri} + \sum_{k=1}^{m_i} \delta_{ik} \, \varepsilon_{ik} \tag{51}$$

M_i = the total mass of link i

$r_{ri} = [\, 1 \;\; r_{xi} \;\; 0 \;\; 0 \,]$, a vector to the center of gravity from joint i (undeformed)

$$\varepsilon_{ik} = \int_0^{l_i} \mu \, [0 \;\; x_{ik} \;\; y_{ik} \;\; z_{ik} \,]^T \, d\eta . \tag{52}$$

Note that ε_{ik} is found in the top row of C_{ik}. It is the distance from the undeformed center of gravity to the center of gravity when all δ are zero except δ_{ik} , which is one. The total distance to the center of gravity from O_i (joint i) is multiplied by the mass to give r_i.

Upon taking the partial derivatives required by Lagrange's equations we find for the joint variables

$$\frac{\partial q_j}{} \quad \sum_{i=j} \frac{}{\partial q_j}$$

For the deflection variables, for $1 \leq j \leq n-1$

$$3V, \quad \ldots \quad A \quad . \quad 3W. \quad x \quad - g^T W_j \epsilon_{jf} . \tag{54}$$

For $j = n$

$$\frac{\partial V_g}{\partial \delta_{nf}} = - g^T W_n \epsilon_{nf} . \tag{55}$$

5. Lagrange's Equations in Simulation Form

At this juncture the components of the complete equations of motion in Lagrangc's formulation, except for the external forcing terms, have been evaluated in equations 38, 47, and 53 for the joint equations; and in equations 39, 48, 54 and 55 for deflection equations. The external forcing terms arc the generalized forces corresponding to the generalized coordinates: the joint and deflection variables in this case. The generalized force corresponding to joint variable q_4 is the joint torque F_i. For the deflection variables the corresponding generalized force will be zero if the corresponding modal deflections or rotations have no displacement at those locations where external forces arc applied. Thus it is assumed for the present development that the modal functions arc selected so that is the case. This is convenient for using the results as well. All motion at the joint is described in terms of the joint variable. (This is not tmc in the approach taken by Sunada and Dubowski [31].) The form of Lagrange's equations will then be:

Hie joint conation j

$$\frac{d}{dt}(3K/3\dot{q}_j)^x - 3K/3q_j + \frac{9V}{dq_j}f + \frac{9V_f}{3q_j} = F_j . \tag{56}$$

The deflection equation j,f

$$\frac{d}{dt}(3K/3\dot{S}_{Jf})^x - 3K/38_{Jf} + \frac{3V}{35_{jf}}* + \frac{3V}{35_{jf}}* = 0 . \tag{57}$$

These equations are in the "inverse dynamic" form. To convert them to the simulation form one must extract the coefficients of the second derivatives of the generalized coordinates to compose an inertia matrix for the system. The second and first derivatives together make up the derivative of the state vector, which can be used in one of the available integration schemes, e.g. Runga-Kutta, to solve for the state as a function of time for given initial conditions and inputs F_i.

5.1. Kinematics Revisited

The purpose of this section will be to extend the kinematics to separate the second derivatives of the joint variables and deflection variables from the expressions for $\ddot{W}_i$ and $\hat{\ddot{V}}_i$. Other occurrences of these derivatives arc already explicit in the formulation as it exists.

First consider the product of transformations which make up W_i and two alternative ways of expressing

$$\hat{W}_i = A_1 E_1 A_2 E_2 \ldots A_h E_h \ldots A_i E_i$$

$$= \hat{W}_{h-1} A_h {}^h\overline{W}_i \tag{58}$$

$$= W_h E_h {}^h\hat{W}_i \tag{59}$$

Carrying through the derivatives one obtains

$$\ddot{\hat{W}}_i = \sum_{h=1}^{i} \left(\hat{W}_{h-1} U_h {}^h\overline{W}_i \ddot{q}_h + \sum_{k=1}^{m_h} W_h M_{hk} {}^h\hat{W}_i \ddot{\delta}_{hk} \right) + \ddot{\hat{W}}_{vi} . \tag{60}$$

For the corresponding expression for W_i write

$$W_i = A_1 E_1 A_2 E_2 \ldots A_h E_h \ldots E_{i-1} A_i$$

$$= \hat{W}_{h-1} A_h {}^h\widetilde{W}_h \tag{61}$$

$$= W_h E_h {}^h W_i . \tag{62}$$

$$\ddot{W}_i = \sum_{h=1}^{i} \hat{W}_{h-1} U_h {}^h\widetilde{W}_i \ddot{q}_h + \sum_{h=1}^{i-1} \sum_{k=1}^{m_h} W_h M_{hk} {}^h\hat{W}_i \ddot{\delta}_{hk} + \ddot{W}_{vi} . \tag{63}$$

The value of $\ddot{\hat{W}}_{vi}$ and $\ddot{W}_{vi}$ can be calculated recursively as shown in equations 15 and 10, respectively, for $\ddot{\hat{W}}_i$ and $\ddot{W}_i$ by only eliminating terms involving $\ddot{q}_j$ and $\ddot{\delta}_{jk}$. The result is

$$\ddot{W}_{vj} = \ddot{\hat{W}}_{v,j-1} A_j + 2 \dot{\hat{W}} \dot{A}_j + \hat{W}_{j-1} U_{2j} \dot{q}_j^2 \tag{64}$$

$$\ddot{\hat{W}}_{vj} = \ddot{W}_{vj} E_j + 2 \dot{W}_j \dot{E}_j . \tag{65}$$

5.2. Inertia Coefficients

To obtain the inertia coefficients that multiply the second derivatives, substitute equations 63 and 60 into the relevant parts of the equations of motion, equations 38 and 39, respectively. Collecting the terms and arranging them for efficient computation requires the steps outlined in this section.

5.2.1. Inertia Coefficients of Joint Variables in the Joint Equations

All occurances of $\ddot{q}_j$ in equation 38 are in the expression for $\ddot{W}_i^T$. When these terms are isolated, a double summation over the indices i and h exists. Interchange the order of the summation as follows:

$$\sum_{i=j}^{n} \sum_{h=1}^{i} = \sum_{h=1}^{n} \sum_{i=\max(h,j)}^{n} .$$

The resulting coefficient for joint variable q_h in the joint equation j is where

$$ {}^{j}\tilde{F}_h = \sum_{i=\max(h,j)}^{n} {}^{j}\tilde{W}_i G_i {}^{h}\tilde{W}_i^T \quad (67) $$

Note that if one exchanges j and h and transposes inside the trace operation an identical expression is obtained. This indicates the symmetry of the inertia matrix which is used to reduce the number of computations required. The expression for ${}^{j}\tilde{F}_h$ can be computed recursively; this will be described later to further improve the efficiency of calculation.

5.2.2. Inertia Coefficients of the Deflection Variables in the Joint Equations

The deflection variables appear both in the expression for $\ddot{W}_i^T$ and explicitly in equation 38. After substituting $\ddot{W}_i^T$ into equation 38, collect terms in $\ddot{\delta}_{jf}$ and exchange the order of summations as follows

$$ \sum_{i=j}^{n} \sum_{h=1}^{i-1} = \sum_{h=1}^{n-1} \sum_{i=\max(h+1,j)}^{n} $$

The resulting coefficient of $\ddot{\delta}_{hk}$ in joint equation j is J_{jhk}. The terms to be included depend on the relative values of j and h. The following hold for $1 \leq k \leq m_h$.

For h = n, j = 1 ... n:

$$ J_{jnk} = 2\,\mathrm{Tr}\left\{ \left(\hat{W}_{j-1} U_j \right) {}^{j}\tilde{W}_n D_{nk} W_n^T \right\}; \quad (68) $$

for h = j ... n-1, j = 1 ... n-1:

$$ J_{jhk} = 2\,\mathrm{Tr}\left\{ \left(\hat{W}_{j-1} U_j \right) \left[{}^{j}F_h M_{hk}^T + {}^{j}\tilde{W}_h D_{hk} \right] W_h^T \right\}; \quad (69) $$

for h = 1 ... j-1, j = 2 ... n:

$$ J_{jnk} = 2\,\mathrm{Tr}\left\{ \left(\hat{W}_{j-1} U_j \right) {}^{j}F_h M_{hk}^T W_n^T \right\}; \quad (70) $$

where for h = 1 ... n-1, j = 1 ... n:

$$ {}^{j}F_h = \sum_{i=\max(h+1,j)}^{n} {}^{j}\tilde{W}_i G_i {}^{h}W_i^T . \quad (71) $$

It can be shown that the inertia coefficient for the deflection variable δ_{hk} in the joint equation j is the same as the coefficient for the joint variable q_j in the deflection equation h,k. This further extends the symmetry of the inertia matrix and reduces the necessary computation.

5.2.3. Inertia Coefficients of the Deflection Variables in the Deflection Equation

In a manner similar to the previous two types of coefficients, the inertia coefficients of the deflection variables in the deflection equations are evaluated. Symmetry of the coefficients can be shown such that the coefficient of variable h,k in equation j,f is the same as the coefficient of variable j,f in equation h,k. Substituting equation 63 into equation 39, isolating the second derivatives of the deflection variables, and interchanging the order of summations enables the inertia coefficients to be identified. Further simplification

$\mathrm{Tr}\{A\ I)\ C\} = \mathrm{Tr}\{C\ A\ B\} = \mathrm{Tr}\{B\ C\ A\}$.

Furthennore the rotation matrices in the transformation matrices arc orthogonal so that $R_i R_i^{T} = I$, a 3x3 identity matrix. This coupled with the zero first row and column of C_{jkf} results in an especially simple form for two of the four cases. The following hold for $1 \leq k \leq m_h$ and $1 \leq f \leq m_j$.

Forj = h = n:

$$I_{nfnk} = 2\,\mathrm{Tr}\{H\ C_{nkf}\} \tag{72}$$

Korj = h =1 ... n-1:

$$I_{jfjk} = 2\,\mathrm{Tr}\{M_{jf}\,{}^j\Phi_j\,M + C_{jkf}\}\ . \tag{73}$$

Forh = n; j = 1 ... n-1:

$$I_{jfnk} = 2\,\mathrm{Tr}\{W_j\,M_{jf}\,{}^jW_n\,D]_{nk}\,W\ \}\ . \tag{74}$$

Forj = 1 ... n-1; h = j+1 ... n-1:

$$I_{j(hk} = 2\mathrm{Tr}\{M_{jr}C^J 4»_h+Mi+\%_h l>_h Jwj;\}. \tag{75}$$

Terms in the above defined forj = 1 ... n-1; h = 1 ... n-1 arc:

$$ {}^j\Phi_h = \sum_{i=\max(j-f\ 1,h-h1)}^{n} HVjG\hat{}W\hat{}. \tag{76}$$

5.2.4. Recursions in the Calculation of the Inertia Coefficients

Since the inertia matrix is a square matrix it requires the calculation of n^ terms where n_t is the total number of variables:

$$n_t = n + \sum_{i=1}^{n} m_i$$

TTic fact that the matrix is symmetrical reduces the number of distinct terms to $n_t(n_L+1)/2$, which still has a second power dependence. Thus while the inverse dynamics computation complexity can be made linear in n_t simulation requires the inertia matrix with complexity dependent on n^. Since n_t can be quite large for practical arms it is important to reduce the coefficient of die squared term as much as possible. Due to their short or even zero length, it is possible for some links to be essentially rigid. Anthropomorphic arms, for example, have two links which arc much longer than the others and tend to dominate the compliance. Many of the terms derived above may not be needed for these links, four of the six links in the anthropomorphic example. Any recursive scheme for calculating the terms in the equations should not require these calculations as a means to get to needed terms.

Consider the calculation of equations 67/71, and 76. Several recursive schemes could be arranged for the efficient calculation of these quantities. Hquation 71 is only needed if the link corresponding to the variable, link h, is flexible. That is, if $m_h > 0$. Iiquation 76 is only needed if both the link of the variable and the link of the equation, link j, is also flexible. Thus we propose the following recursive scheme for calculating Γ_v, F_h and Φ_h. The following hold for $1 \le h \le n_v$, $1 \le j \le m_j$:

Initialization:

$$^{n}\tilde{F}_{n} = G_{n}. \tag{77}$$

For $j > h \le n$:

$$^{j}\tilde{F}_{h} = E_{j} \Lambda_{j} \, ^{j+1}\tilde{F}_{h} \tag{78}$$

For $j = h$:

If $m_h > 0$ calculate:

$$^{j}F_{h} = \, ^{j}\tilde{F}_{h+1} \, \Lambda_{vh}^{T}. \tag{80}$$

If $m_h > 0$ and $m_j > 0$ calculate:

$$^{j}\Phi_{h} = \Lambda_{j+1} \, ^{j}\tilde{F}_{h}. \tag{81}$$

5.3. Assembly of Final Simulation Equations

The complete simulation equations have now been derived. It remains to assemble them in final form and to point out some remaining recursion relations that can be used to reduce the number of calculations. The second derivatives of the joint and deflection variables arc desired on the "left hand side" of the equation as unknowns and the remaining dynamic effects and the inputs arc desired on the "right hand side." To carry out tiiis process completely one would take the inverse of the inertia matrix J and prcmultipty the vector of other dynamic effects. This inverse can only be evaluated numerically because of its complexity. Thus for the present purposes die equations will be considered complete in die following form:

$$J \ddot{z} = R, \tag{82}$$

where

J = Inertia matrix consisting of coefficients previously defined in die order for multiplication appropriate for z

z = die vector of generalized coordinates
 $= [q_1 \; \delta_{11} \; \delta_{12} \; .... \; \delta_{1m_1} \; q_2 \; \delta_{21} ... \delta_{2m_2} .... q_h \; \delta_{h1} ... \delta_{hk} ... \delta_{hm_h} ..... \delta_{nm_n}]^T$

q_h = the joint variable of the h-th joint

δ_{hk} = die deflection variable (amplitude) of die k-thmodc of link h

R = vector of remaining dynamics and external forcing terms
 $= [R_1 \; R_{11} \; R_{12} \; ... \; R_{1m_1} \; R_2 \; R_{21} \; ... \; R_{2m_2} \; ... \; R_j \; R_{j1} \; ... \; R_{jf} \; ... \; R_{jm_j} \; ... \; R_{nm_n}]^T$

R_j = dynamics from the joint equation j (equation 56) excluding second derivatives of the generalized coordinates

R_{jf} = dynamics from the deflection equation jf (equation 57) excluding second derivatives of the generalized coordinates

The elements of **J** have just been formulated and can be arranged to form the proper equations in the order described above. This order has been selected because it results in the symmetrical appearance of **J**. The elements of **R** have not been explicitly given with the second derivatives removed. These are given below with some recursions to facilitate their computation.

$$R_1 = -2\,\mathrm{Tr}\left\{ U_1 Q_1 \right\} + g^T U_1 P_1 + F_1 \tag{83}$$

$$R_j = -2\,\mathrm{Tr}\left\{ \hat{W}_{j-1} U_j Q_j \right\} + g^T \hat{W}_{j-1} U_j P_j + F_j \tag{84}$$

$$R_{nf} = -2\,\mathrm{Tr}\left\{ \left[\ddot{W}_{vn} D_{nf} + 2\dot{W}_n \sum_{k=1}^{m_n} \dot{\delta}_{nk} C_{nkf} \right] W_n^T \right\} - \sum_{k=1}^{m_n} \delta_{nk} K_{nkf} + g^T W_n \varepsilon_{nf} \tag{85}$$

$$R_{jf} = -2\,\mathrm{Tr}\left\{ W_j M_{jf} A_{j+1} Q_{j+1} \left[\ddot{W}_{vj} D_{jf} + 2\dot{W}_j \sum_{k=1}^{m_j} \dot{\delta}_{jk} C_{jkf} \right] W_j^T \right\}$$

$$- \sum_{k=1}^{m_j} \delta_{jk} K_{jkf} + g^T W_j M_{jf} A_{j+1} P_{j+1} + g^T W_j \varepsilon_{jf} \tag{86}$$

where

$$Q_n = G_n \ddot{W}_{vn}^T + 2\left(\sum_{k=1}^{m_n} \dot{\delta}_{nk} D_{nk} \right) \dot{W}_n^T \tag{87}$$

$$Q_j = G_j \ddot{W}_{vj}^T + 2\left(\sum_{k=1}^{m_j} \dot{\delta}_{jk} D_{jk} \right) \dot{W}_j^T + E_j A_{j+1} Q_{j+1} \tag{88}$$

$$P_n = M_n r_n + \sum_{k=1}^{m_j} \delta_{nk} \varepsilon_{nk} \tag{89}$$

$$P_j = M_j r_j + \sum_{k=1}^{m_j} \delta_{jk} \varepsilon_{jk} + E_j A_{j+1} P_{j+1} \tag{90}$$

6. Conclusions

The above model is successful in terms of its accuracy and its speed. The two qualities are somewhat related in that accuracy of the flexible representation can be improved by increasing the number of modes used to represent the link deflection at the expense of calculation time. The issue is further complicated by the choice of mode shapes, range of motion considered, and the arm configuration. Furthermore, limited information is available in the literature for comparison. A simple comparison has been used in the past and can be performed for calculation complexity. Hollerbach [15] compares several approaches to the inverse dynamics problem of rigid arms by different authors. Walker [33] gives a similar count for four approaches to the simulation problem. Sunada [31] has given computation times for a given manipulator, trajectory, and computer for his flexible simulation. Comparison to the calculation counts of rigid models are given for a rough comparison of speeds in this section. No attempt at a quantitative comparison of the accuracy is made.

To determine the number of calculations from the equations, a choice must be made on how some matrix products are implemented. Hollerbach chose to use the most straightforward implementation of the equations. The approach here is quite different. Obvious simplifications in the multiplication of matrices with known constant rows, the top row of a transformation matrix for example, are assumed in these computations. The 4x4 matrix transformation was chosen for its conceptual convenience and the calculation count will not be intentionally penalized for that choice. Furthermore, certain products appear in multiple equations and are assumed to be saved when needed later. Special purpose multiply routines are used whenever they can capitalize on the special structure of a given matrix. Finally, in the simulation form the calculations needed to invert the inertia matrix are not included, and no consideration is given to the calculations of the integration routine. The general form of the modal parameters are used however. This results in all combinations of modes h and k in the matrix C_{ihk} to be computed and used and hence introduces a squared dependence on the number of modes on *each* inertial coefficient of the deflection variables. With these assumptions the number of calculations is approximate:

<u>Number of multiplications:</u>

$$6\, n_f^2 m^2 + 17.5\, n_f m^2 + 118\, n_f^2 m + 74\, n\, n_f m +$$
$$137.5\, n_f m + 84\, n^2 + 86\, n\, n_f + 279\, n + 126\, n_f - 57$$

Number of additions:

$$6.5\, n_f^2 m^2 + 19 n_f m^2 + 115.5\, n_f^2 m + 68\, n n_f m +$$
$$123\, n_f m + 85\, n^2 + 80\, n\, n_f + 329\, n + 111\, n_f - 91$$

where: n = total number of joints
n_f = number of flexible links
m = number of modes describing each flexible link

The above approximation assumes an "average" joint complexity over two common types of rotary joints, the same number of modes on each flexible link, a rigid last link and a flexible first link.

If assumed mode shapes arc restricted so that the shape functions in the x, y, and z directions are orthogonal, only C_{ikk} will be non-zero. This is a stronger requirement than the orthogonality of the set of complete mode shapes, but would often be realized with simple mode shapes. It has not been determined if this would improve the combination of speed and accuracy.

This calculation count can be roughly compared to rigid link results available in the literature mentioned above. For a 12 degree of freedom rigid problem die inverse 3x3 transformation matrix formulation requires 2.66 times as many multiplies as the Newton-Kulcr formulation. Walker's method 3 (his best) for simulation requires 4,491 multiplies. For 6 joints, and two flexible links with 3 modes each the method of this paper requires approximately 12,009 multiplies. The ratio of these simulation methods is 2.67, almost exactly the same as for the inverse dynamic methods with the same number of degrees of freedom. A modal representation of flexibility would be much more accurate than adding 6 imaginary joints to represent compliance, but one could expect to use 15 imaginary joints and 6 real joints with Walker's method with fewer multiplies than with the method of this paper.

Thus it seems that in order to be competitive with possible Newton-Ruler, non-transfer matrix approaches, the simplification of the assumed mode shapes will have to be made. It is not clear that the conceptual convenience of the transformation matrix approach can be justified relative to vector dyadic approaches of Hughes [17] and Likins[28]. Unfortunately, computation counts are not available for that work.

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