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A SEMANTIC CHARACTERIZATION OF FULL ABSTRACTION

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1. Introduction

Semantic characterization of full abstraction has been a well known open problem for many years. It arises in the context of the Scott-Strachey denotational approach to semantics. In this approach each programming construct is given a denotation in a mathematical model. Of course, if the semantics is to be of any use at all, it must have the property that whenever two constructs have same denotations they must behave identically in all programming contexts. However, the converse is difficult to ensure. This demands that two programming constructs have the same denotations whenever they behave identically in all programming contexts. This is the well known problem of full abstraction. For a special case of typed lambda calculus, PCF, it was shown by Plotkin that the classical model consisting of domains of continuous functions is not fully abstract. However, he was able to make the model fully abstract by adding to the language a new programming construct which provided a *parallel or* facility. On the other hand Milner was able to obtain a *syntactic* fully abstract model for the typed lambda calculus. However the semantic characterization of full abstraction remained an open problem. A satisfactory semantic characterization should construct a fully abstract model in a *semantic* way and should point·out its relationship with the original model. In this paper we provide such a characterization. We construct an extensional, fully abstract and algebraic model for the typed lambda calculus which turns out to be just a retract of the original model. Moreover the new denotational semantics decomposes nicely in the sense: the new denotation of a term turns out to be just a homomorphic retraction of its old denotation. This, we hope, provides a very satisfactory semantic characterization of full abstraction. Moreover the theory can be extended to take into account the presence of the reflexive (i.e. recursively defined) types.

2. Typed Lambda Calculus

We assume some familiarity with typed lambda calculus and combinators.

Assume we are given a set of ground types. We shall let κ, κ_1, $\kappa_2 \ldots$ range over these ground types. From the given ground types we define the set of functional types, Γ:

1. $\kappa \in \Gamma$, if κ is a given ground type,
2. $(\alpha \rightarrow \beta) \in \Gamma$ whenever $\alpha, \beta \in \Gamma$.

We shall omit ( ) whenever possible. In this case the association of ( ) is taken as from right to left. Thus $\alpha \rightarrow \beta \rightarrow \sigma$ denotes $\alpha \rightarrow (\beta \rightarrow \beta)$. We write:

$$(\sigma_1, \ldots, \sigma_n, \tau) = \sigma_1 \rightarrow \cdots \rightarrow \sigma_n \rightarrow \tau \text{ for } n \geq 0.$$

Note that each type can be written in the form $(\sigma_1, \ldots, \sigma_n, \kappa)$. A type is said to be of first order if it is of the form $(\kappa_1, \ldots, \kappa_n, \kappa)$. We assume that we are given, for each ground type κ, a set basic constants $\mathcal{B}^\kappa$ and for each first order type τ a set of basic function constants $\mathcal{F}^\tau$. A family $\{\mathcal{L}^\tau\}$, where $\mathcal{L}^\tau$ is intended to be a set of terms of type τ, is defined to be the family of the smallest sets $\mathcal{L}^\tau$ satisfying the conditions:

1. $\mathcal{B}^\kappa \subseteq \mathcal{L}^\kappa$, for all ground types κ.
2. $\mathcal{F}^\tau \subseteq \mathcal{L}^\tau$, for all first order types τ.
3. $s \in \mathcal{L}^\tau$ where S is the usual closed combinator $\lambda xyz.(xz)(yz)$ of type τ.
4. $K \in \mathcal{L}^\tau$ where K is the usual closed combinator $\lambda xy.x$ of type τ.
5. $Y \in \mathcal{L}^\tau$ where Y, which is intended to be a fixed point combinator, is of type τ.
6. $(t\ s) \in \mathcal{L}^\beta$ whenever $t \in \mathcal{L}^{\alpha \rightarrow \beta}$ and $s \in \mathcal{L}^\alpha$.

We shall omit the braces ( ) whenever possible, the association then being from left to right. Thus we shall write ts instead of (ts). Also we let Ω denote the combinator $Y(\lambda x.x) = Y(SKK)$. Ofcourse Ω is the usual 'undefined' term.

Define $\mathcal{L} = \bigcup \mathcal{L}^\sigma$, $\mathcal{F} = \bigcup \mathcal{F}^\tau$, where τ ranges over the first order types, and $\mathcal{B} = \bigcup \mathcal{B}^\kappa$.

We now give an operational semantics to the above language. Assume that we are given, for every first order function constant $f : (\kappa_1, \ldots, \kappa_n, \kappa)$, a reduction rule $\xrightarrow{f}$, which is a *partial* function from $\mathcal{L}^{\kappa_1} \times \cdots \times \mathcal{L}^{\kappa_n}$ to $\mathcal{B}^\kappa$. If $\xrightarrow{f} (t_1, \ldots, t_n) = b$, we represent this pictorially as:

$$f t_1 \ldots t_n \xrightarrow{f} b.$$

We assume that $\xrightarrow{f}$ is total on $\mathcal{B}^{\kappa_1} \times \cdots \times \mathcal{B}^{\kappa_n}$. This means $\xrightarrow{f}$ reduces every term of the form $f b_1 \ldots b_n$, where $b_i \in \mathcal{B}^{\kappa_i}$, to some basic constant $b \in \mathcal{B}^\kappa$. Pictorially:

$$f b_1 \ldots b_n \xrightarrow{f} b.$$

This restriction is not severe because one can assume that each $\mathcal{B}^\kappa$ contains the 'undefined' constant. When f is a 'call by value' function then the domain of the function $\xrightarrow{f}$ will be $\mathcal{B}^{\kappa_1} \times \cdots \times \mathcal{B}^{\kappa_n}$, because, in that case, f insists that its arguments be evaluated completely before it is called. But this need not be the case always. For example, f might choose to ignore its second argument sometimes.

Now we define a reduction relation $\longrightarrow$ on terms as follows:

1. $\dfrac{f t_1 \dots t_n \xrightarrow{f} b}{f t_1 \dots t_n \longrightarrow b}$, where $t_j \in \mathcal{L}^{\kappa_j}$, $b \in \mathcal{B}$ and $f \in \mathcal{F}$.
2. $(Y\, f) \longrightarrow f\,(Y\, f)$, where $f \in \mathcal{L}^{\sigma \to \sigma}$ and $t \in \mathcal{L}^{\sigma}$.
3. $S\, r\, s\, t \longrightarrow (r\, t)\,(s\, t)$,
4. $K\, r\, s \longrightarrow r$,
5. $\dfrac{r \longrightarrow r'}{r\, s \longrightarrow r'\, s}$,
6. $\dfrac{s \longrightarrow s'}{r\, s \longrightarrow r\, s'}$.

We shall denote by $\xrightarrow{*}$ the transitive reflexive closure of $\longrightarrow$.

We assume that $\xrightarrow{f}$ is reasonable, i.e. it satisfies the following consistency constraint:

$$\text{if } f t_1 \dots t_n \xrightarrow{f} b \text{ and } t_j \xrightarrow{*} s_j \text{ for all } j \text{ then } f s_1 \dots s_n b.$$

We know that Church-Rosser theorem holds for pure typed lambda calculus. The above consistency constraint ensures that it holds for $\mathcal{L}$ too. Hence, though $\longrightarrow$ is not monogenic, it has a unique normal form –if the normal form exists.

We turn next to the denotational semantics of $\mathcal{L}$. Assume that we are given, for each ground type κ, a ground domain D^{κ} and a type-respecting ground semantics $G : \mathcal{B} \to \bigcup D^{\kappa}$ such that all the finite elements of the ground domains are definable by basic constants, i.e, for each finite $d \in D^{\kappa}$ there exists a basic constant $b \in \mathcal{B}^{\kappa}$ such that $d = G\, b$. We also assume that for each $f \in \mathcal{F}^{\tau}$, where $\tau = (\kappa_1, \dots, \kappa_n, \kappa)$, we are given a first order continuous function $H f$ such that for all $b_1, \dots, b_n \mathcal{B}$,

$$(H f)(G b_1) \dots (G b_n) = (G b) \text{ if } f b_1 \dots b_n \xrightarrow{f} b.$$

Note that, as all the finite elements of ground domains are assumed to be definable by the basic ground constants, the above condition uniquely determines $H f$.

A model of $\mathcal{L}$, $M = (D^{\tau}, \cdot, A)$, consists of

1. a cpo D^{τ} for each type τ such that for each ground type κ, D^{κ} is isomorphic to the given ground domain. (We shall assume this isomorphism implicitly and not refer to it explicitly.)
2. a continuous application function $\cdot : D^{\alpha \to \beta} \times D^{\alpha} \to D^{\beta}$ for all types α and β.
3. a type preserving map $A : \mathcal{L} \to \bigcup D^{\tau}$ which is a homomorphism:

$$A(t\, s) = (A t) \cdot (A s) \quad \text{for all } t, s \in \mathcal{L}.$$

Again we shall omit $\cdot$ whenever possible, the association being assumed to be from left to right. We shall denote by A^{τ} the restriction of A to $\mathcal{L}^{\tau}$. This definition of model is akin to that of a *general interpretation* in [Plotkin].

A model, $M = (D^\tau, \cdot, A)$, is called standard if

1. $A\,b = G\,b$, for all $b \in \mathcal{B}$, i.e. A agrees with G on the ground constants.
2. $Af = Hf$ if $f \in \mathcal{F}$.

If $b \in \mathcal{B}$ and $f \in \mathcal{F}$, we shall ambiguously use the symbol b to denote $G(b)$ and the symbol f to denote Hf. Whether a symbol b or f is playing a syntactic role or a semantic one should be clear from the context.

If a model for $\mathcal{L}$ is to be of any value, it should be faithful to its operational semantics. Let us define a type-respecting map $O : \bigcup \mathcal{L}^\kappa \to \bigcup D^\kappa$ as:

$$O\,t = \begin{cases} G\,b & \text{if } t \xrightarrow{*} b \\ \bot & \text{otherwise, i.e. if all computation of } t \text{ diverge.} \end{cases}$$

Note that because a normal form, if it exists, is unique by Chuch-Rosser theorem, O is well defined.

A model, $M = (D^\tau, ., A)$, is called adequate (or faithful) if, for all ground terms t,

$$A\,t = O\,t.$$

In this case we say that O and A are semantically equivalent.

One can now define precisely, what it means to say that one term is operationally weaker than the other.

We say $t \overset{\mathcal{L}}{\underset{\sim}{\sqsubseteq}} s$, where t, $s \in \mathcal{L}^\sigma(\sigma_1, \ldots, \sigma_n, \kappa)$, if for all $t_i \in \mathcal{L}^{\sigma_i}$:

$$O(t\,t_1\,\ldots\,t_n) \sqsubseteq O(s\,t_1\,\ldots\,t_n). \tag{1}$$

This definition differs from the usual definition found in literature which is given in terms of 'contexts'. We shall see later that the two definitions are equivalent.

A model, $M = (D^\tau, ., A)$, is called fully abstract if for all terms t, $s \in \mathcal{L}^\tau$:

$$A\,t \sqsubseteq A\,s \text{ iff } t \overset{\mathcal{L}}{\underset{\sim}{\sqsubseteq}} s.$$

A simple model for $\mathcal{L}$ is the classical model, $M = (D^\tau, \cdot, A)$, where the ground domains are the given ones and domains at higher types are inductively defined simply as follows: $D^{\alpha \to \beta} = D^\alpha \to D^\beta$, where $D^\alpha \to D^\beta$ is the domain of continuous functions from D^α to D^β. The application function $\cdot$ is the usual function application and A is defined as follows:

1. $A(b) = b$, for $b \in \mathcal{B}$
2. $A(f) = f$, for $f \in \mathcal{F}$
3. $A(S) = \lambda xyz.\,(xz)(yz)$
4. $A(K) = \lambda xy.\,x$
5. $A(Y) = \bigsqcup_{n=0}^{\infty} \lambda f.\,f^n(\bot)$
6. $A(t\,s) = (At)\,(As)$.

It can be shown that M is an adequate model of $\mathcal{L}$ (for a similar proof see [Plotkin]). However, it was shown by Plotkin that M is not fully abstract for a special case of $\mathcal{L}$ PCF. But in this case Plotkin made M fully abstract by adding an extra *parallel or* facility to $\mathcal{L}$. On the other hand, Milner demonstrated the existence of fully abstract model for $\mathcal{L}$ in [Milner]. However, his construction is completely syntactic and his model doesn't seem to have any obvious relationship with the model M. Many attempts have been made to construct such a model in a semantic way, especially noteworthy being those of Berry and his colleagues. (See [Berry] which gives an excellent history of the problem and an extensive list of bibliographical references.) Nevertheless the semantic characterization of full abstraction remained an important open problem. We provide that semantic characterization in this paper. Our final model will pleasingly turn out to be a submodel of M.

It should be made clear at the outset that we dealing with general typed lambda calculi as in [Milner]. In particular, we do not assume that $\mathcal{L}$ has any *addtional* properties like sequentiality (see [Berry]). Our method will provide a semantic characterzation of full abstraction for any general typed lambda calculus, but we shall leave it open whether a *better* semantic characterization can be found when the language $\mathcal{L}$ is known to have the above additional properties.

Henceforth M will refer to the above-mentioned classical model $(D^\tau, \cdot, A)$. We shall also assume that D^τs are ω-algebraic complete lattices. Why we require D^τ's to be complete lattices and not simply consistently complete cpos will become clear in the next section.

3. Inclusive Predicates

The collapsing of M onto a fully abstract model is achieved through some inductively defined inclusive predicates (see [Milne], [Mulmuley], [Reynolds]). For each type τ we define an inclusive predicate $\Theta^\tau \subseteq D^\tau \times \mathcal{L}^\tau$ as follows:

1. For a ground type κ,
$$\Theta^\kappa = \{(d,t) \mid d \sqsubseteq O(t)\}.$$

2. For a type $\tau = \alpha \to \beta$,
$$\Theta^\tau = \{(d,t) \mid \forall (c,s) \in \Theta^\alpha.(dc,ts) \in \Theta^\beta\}.$$

It is easy to show that Θs can be defined equivalently as follows:

1. For a ground type κ,
$$\Theta^\kappa = \{(d,t) \mid d \sqsubseteq O(t)\}.$$

2. For a type $\tau = (\sigma_1, \ldots, \sigma_n, \kappa)$
$$\Theta^\tau = \{(d,t) \mid \forall (d_i,t_i) \in \Theta^{\sigma_i}.dd_1 \ldots d_n \sqsubseteq O(tt_1 \ldots t_n)\}.$$

We shall use any of the two equivalent formulations as convenient. It is easy to show that all Θ^τs are directed complete. Note that $(d,e) \in \Theta$ can be taken as saying d *is weaker than* e *in some sense* . Hence Θ^τ can be used to define a natural quasiorder $\underset{\sim_\tau}{\sqsubseteq}$ on D^τ. We say

$$d_1 \underset{\sim_\tau}{\sqsubseteq} d_2 \text{ iff for all } t,\ (d_2,t) \in \Theta^\tau \text{ implies } (d_1,t) \in \Theta^\tau.$$

Let $\simeq_\tau$ be the induced equivalence relation. The equivalence class of $d \in D^\tau$ will be denoted by $[d]^\tau$.

In this paper we adopt the convention of dropping the type subscripts and superscripts whenever no ambiguity arises. Thus we shall often write $\underset{\sim}{\sqsubseteq}$, $\simeq$, $[\ ]$, or Θ instead of $\underset{\sim_\tau}{\sqsubseteq}$, $\simeq_\tau$, $[\ ]^\tau$, or Θ^τ. The convention also applies to any definitions we introduce in future.

The inclusive relation Θ and the induced equivalence relation $\simeq$ have many nice properties. For example,

$$\text{if } t \overset{\mathcal{L}}{\underset{\sim}{\sqsubseteq}} s \text{ then } (d,t) \in \Theta \text{ implies } (d,s) \in \Theta.$$

Secondly,

$$\sqsubseteq \text{ is a refinement of } \underset{\sim}{\sqsubseteq}, \text{ i.e, if } d_1 \sqsubseteq d_2 \text{ then } d_1 \underset{\sim}{\sqsubseteq} d_2.$$

Let $D^\tau_\simeq$ denote the set of equivalence classes of D^τ. We can convert $D^\tau_\simeq$ into a partial order as follows. We say

$$[d_1] \sqsubseteq [d_2] \text{ iff } d_1 \underset{\sim}{\sqsubseteq} d_2.$$

The choice of the representatives d_1 and d_2 in the above definition is immaterial. Using the directed completeness of Θ it can be shown that $D^\tau_\simeq$ is actually a cpo. In fact, for any chain $\{d_i \mid i \geq 0\}$, $\bigsqcup_{i=0}^{\infty}[d_i] = [\bigsqcup_{i=0}^{\infty} d_i]$. Thus we have a continuous function $[\ ]^\tau : D^\tau \to D^\tau_\simeq$.

As the D^τs are assumed to be complete lattices it is easy to see that whenever $d_1 \in [d]$ and $d_2 \in [d]$ then $d_1 \sqcup d_2 \in [d]$. (This where we need the lattice property of D^τ.) This means that $[d]$ is directed. By the directed completeness of the Θs it follows that each $[d]$ has a maximum element $max[d] = \bigsqcup[d]$.

Let us consider a 'retract' $Q^\tau : D^\tau \to D^\tau_\sim$:

$$Q^\tau : d \mapsto max[d]^\tau.$$

Q^τ is in some sense a representation of the *Quotient Space* $D^\tau_\sim$ (and hence the mnemonic name Q^τ). Let us consider a 'model', $M^Q = (Q^\tau, \cdot, A^Q)$, where $A^Q = Q \circ A$ and . is just the restriction of the application operation in M and hope that M^Q is a fully abstract, extensional model of $\mathcal{L}$. Alas! our attempt is doomed to failure as Q^τ defined above might not be continuous, though it is certainly monotonic and satisfies the retract definition $Q^\tau \circ Q^\tau = Q^\tau$. Even if it were continuous the difficulties do not end. How do we know, for example, that we are justified in letting the application in M^Q be just the restriction of the application in M? That is to say, is it the case that whenever $d \in Q^\sigma\alpha \to \beta$ and $c \in Q^\alpha$ then $dc \in Q^\beta$? And, of course, we have to show that it is indeed fully abstract, extensional, algebraic ...

Definitely we need to be more sophisticated! What we shall do is to consider a sequence of subsets of $\mathcal{L}$, $\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \ldots$. For each $\mathcal{L}_i$ we construct a fully abstract, extensional model M_i^Q. The final model M^Q is then obtained as a *limit* of the sequence $M_1^Q, M_2^Q, \ldots$. Of course, the success of the approach depends on choosing each $\mathcal{L}_i$ wisely. Before we do that we need to extend the notion of a model.

Let $\mathcal{K}$ be a subset of $\mathcal{L}$ which is closed under application, i.e, whenever $t \in \mathcal{K}$ and $s \in \mathcal{K}$ then $ts \in \mathcal{K}$.

A $\mathcal{K}$-model, $N = (E^\tau, \cdot, B)$, consists of

1. a cpo E^τ for each type τ such that, for each ground type κ, E^κ is a subdomain of the given ground domain Dk.
2. a continuous application function $\cdot : E^{\alpha \to \beta} \times E^\alpha \to E^\beta$ for all types α and β.
3. a type preserving map $B : \mathcal{K} \to \bigcup E^\tau$ which is a homomorphism:

$$B(t\, s) = (Bt)\,(Bs) \text{ for } t,\, s \in \mathcal{K}.$$

It is clear that a model for $\mathcal{L}$, as defined in the previous section is just an $\mathcal{L}$-model.

A $\mathcal{K}$-model, $N = (E^\tau, \cdot, B)$, is said to be adequate if $B(t) = O(t)$ for every ground term $t \in \mathcal{K}$.

Let $\mathcal{K}^\tau = \mathcal{L}^\tau \cap \mathcal{K}$. Given $t, s \in \mathcal{K}^{(\sigma_1, \ldots, \sigma_n, \kappa)}$, we say $t \overset{\mathcal{K}}{\underset{\sim}{\sqsubset}} s$, if for all $t_i \in \mathcal{K}^{\sigma_i}$,

$$O(tt_1 \ldots t_n) \sqsubseteq O(st_1 \ldots t_n).$$

We say that a $\mathcal{K}$-model, $N = (E^\tau, \cdot, B)$, is fully abstract if for all $t, s \in \mathcal{K}^\tau$:

$$B(t) \sqsubseteq B(s) \text{ iff } t \overset{\mathcal{K}}{\underset{\sim}{\sqsubset}} s.$$

We can now address the question of selecting the sequence of subsets of $\mathcal{L}$, $\mathcal{L}_1 \subseteq \mathcal{L}_2 \ldots$ Suppose that we are given, for each ground type κ, a monotone sequence of *finite* projections, $\phi_1^\kappa \sqsubseteq \phi_2^\kappa \ldots$ such that $\bigsqcup_{i=0}^{\infty} \phi_i^\kappa = I$, where I is the identity function on D^κ. Finiteness of ϕ_i^κ implies that $|\phi_i^\kappa|$, the fixpoint set of ϕ_i^κ, is finite and moreover each fixpoint of ϕ_i^κ is a finite element of D^κ. For every higher type $\tau = \alpha \to \beta$ we inductively define ϕ_i^τ:

$$\phi_i^\tau = \lambda f : \tau . \phi_i^\beta \circ f \circ \phi_i^\alpha .$$

We shall denote $|\phi_i^\tau|$ by D_i^τ. It follows that each ϕ_i^τ is finite– hence D_i^κ is finite– and also that $\bigsqcup_{i=0}^{\infty} \phi_i^\tau = I$. Also $D_i^{\alpha \to \beta}$ is isomorphic to the function space $D_i^\alpha \to D_i^\beta$.

We make an important assumption.

We assume that each ϕ_i^κ is definable in $\mathcal{L}$.

This means that there exists a term $\Phi_i^\kappa \in \mathcal{L}$ such that $A(\Phi_i^\kappa) = \phi_i^\kappa$. It follows by induction that ϕ_i^τ is definable for every type τ; we let, for $\tau = \alpha \to \beta$,

$$\Phi_i^\tau = \lambda f : \tau . \Phi_i^\beta \circ f \circ \Phi_i^\alpha .$$

(Strictly speaking Φ_i^τ is an S-K combinator equivalent to the right-hand side of the above equation). Then it is easily seen that $A(\Phi_i^\tau) = \phi_i^\tau$.

For each term $t : \tau \in \mathcal{L}$, we define its i^{th} syntactic approximant $\lfloor t \rfloor_i \in \mathcal{L}$ as:

$$\lfloor t \rfloor_i = \Phi_i^\tau \, t .$$

Let $\mathcal{L}_i$ be the smallest set closed under application which contains $\lfloor t \rfloor_i$ for each $t \in \mathcal{L}$. Then

$$A(s) \in D_i^\tau \text{ for every } s : \tau \in \mathcal{L}_i .$$

and moreover

$$A(t) = \bigsqcup_{i=0}^{\infty} A(\lfloor t \rfloor_i) \text{ for every } t \in \mathcal{L}.$$

Let $M_i = (D_i^\tau, \cdot, A_i)$, where $A_i : \mathcal{L}_i \to \bigcup D_i^\tau$ is simply the restriction of A to $\mathcal{L}_i$. Then M_i is an adequate, extensional $\mathcal{L}_i$-model. We shall collapse M_i onto a fully abstract, extensional model M_i^Q. But before that let us investigate the relationship between the operational preorder w.r.t $\mathcal{L}$ and the operational preorder w.r.t $\mathcal{L}_i$.

Lemma 3.1: For all i,

1. if $t, s \in \mathcal{L}$ then $t \overset{\mathcal{L}}{\underset{\sim}{\sqsubset}} s$ implies $\lfloor t \rfloor_i \overset{\mathcal{L}_i}{\underset{\sim}{\sqsubset}} \lfloor s \rfloor_i$.

2. if $t, s \in \mathcal{L}_i$ then $t \overset{\mathcal{L}_i}{\underset{\sim}{\sqsubset}} s$ implies $t \overset{\mathcal{L}}{\underset{\sim}{\sqsubset}} s$.

Proof: 1) Suppose $t, s \in \mathcal{L}$ and $t \overset{\mathcal{L}}{\underset{\sim}{\sqsubset}} s$. Then for all $t_1, \ldots, t_n \in \mathcal{L}_i$ we have, as $\mathcal{L}_i \subseteq \mathcal{L}$,

$$O(t t_1 \ldots t_n) \sqsubseteq O(s t_1 \ldots t_n).$$

Hence,

$$\phi_i^\kappa(O(tt_1 \dots t_n)) \sqsubseteq \phi_i^\kappa(O(st_1 \dots t_n)) \tag{2}$$

But,

$$\begin{aligned}
\phi_i^\kappa(O(tt_1 \dots t_n)) &= \phi_i^\kappa(A(tt_1 \dots t_n)) && \text{by the semantic equivalence between } O \text{ and } A. \\
&= \phi_i^\kappa((At)(At_1) \dots (At_n)) \\
&= \phi_i^\kappa((At)(\phi_i^{\sigma_1} \circ A(t_1)) \dots (\phi_i^{\sigma_n} \circ A(t_n))) && \text{because, for all } j,\ t_j \in \mathcal{L}_i \text{ and hence } A(t_j) \in D_i^{\sigma_j} \\
&= (\phi_i^\tau \circ A(t))(At_1) \dots (At_n) \\
&= (A\lfloor t \rfloor_i)(At_1) \dots (At_n) \\
&= A(\lfloor t \rfloor_i t_1 \dots t_n) \\
&= O(\lfloor t \rfloor_i t_1 \dots t_n) && \text{by the semantic equivalence.}
\end{aligned}$$

And similarly,

$$\phi_i^\kappa(O(st_1 \dots t_n)) = O(\lfloor s \rfloor_i t_1 \dots t_n).$$

From (2) we conclude that:

$$O(\lfloor t \rfloor_i t_1 \dots t_n) \sqsubseteq O(\lfloor s \rfloor_i t_1 \dots t_n) \text{ for all } t_j \in \mathcal{L}_i.$$

Thus indeed $\lfloor t \rfloor_i \overset{\mathcal{L}_i}{\underset{\sim}{\sqsubseteq}} \lfloor s \rfloor_i$.

2)Assume $t, s : (\sigma_1, \dots, \sigma_n, \kappa) \in \mathcal{L}_i$ and $t \overset{\mathcal{L}_i}{\underset{\sim}{\sqsubseteq}} s$. Then, for all $t_j : \sigma_j \in \mathcal{L}$, a similar calculation yields:

$$\begin{aligned}
O(tt_1 \dots t_n) &= A(tt_1 \dots t_n) \\
&= (At)(At_1) \dots (At_n) \\
&= (At)(\phi_i^{\sigma_1} \circ A(t_1)) \dots (\phi_i^{\sigma_n} \circ A(t_n)) \quad \text{as } A(t) \in D_i^\tau \text{ because } t \in \mathcal{L}_i \\
&= (At)(A\lfloor t_1 \rfloor_i) \dots (A\lfloor t_n \rfloor_i) \\
&= A(t\lfloor t_1 \rfloor_i \dots \lfloor t_n \rfloor_i) \\
&= O(t\lfloor t_1 \rfloor_i \dots \lfloor t_n \rfloor_i)
\end{aligned}$$

That is, for all $t_j : \sigma_j \in \mathcal{L}$,

$$O(tt_1 \dots t_n) = O(t\lfloor t_1 \rfloor_i \dots \lfloor t_n \rfloor_i) \tag{3}$$

And similarly,

$$O(st_1 \dots t_n) = O(s\lfloor t_1 \rfloor_i \dots \lfloor t_n \rfloor_i) \tag{4}$$

As $t \overset{\mathcal{L}_i}{\underset{\sim}{\sqsubseteq}} s$, we conclude that for all $t_j \in \mathcal{L}$

$$O(t\lfloor t_1 \rfloor_i \dots \lfloor t_n \rfloor_i) \sqsubseteq O(s\lfloor t_1 \rfloor_i \dots \lfloor t_n \rfloor_i),$$

because $\lfloor t_j \rfloor_i \in \mathcal{L}_i$. Thus from (3) and (4) we conclude that for all $t_j : \sigma_j \in \mathcal{L}$

$$O(tt_1 \dots t_n) \sqsubseteq O(st_1 \dots t_n),$$

which means $t \overset{\mathcal{L}}{\underset{\sim}{\sqsubseteq}} s$.

$\diamond$

4. A finite approximate model

In this section we describe how one can construct, for each $\mathcal{L}_i$, a model M_i^Q which will be a finite approximation to the final fully abstract $\mathcal{L}$-model, M^Q.

For each $d \in D_i^\tau$ define $[d]_i^\tau = D_i^\tau \cap [d]^\tau$. Then it is easy to see that $[d]_i^\tau$ also has a maximum; $max[d]_i^\tau = \phi_i^\tau(max[d]^\tau)$. Thus we have a map $max_i^\tau : D_i^\tau \to D_i^\tau$,

$$max_i^\tau : d \mapsto max[d]_i^\tau,$$

which is monotonic and hence trivially continuous as D_i^τ is finite. Let

$$Q_i^\tau = max_i^\tau \circ \phi_i^\tau.$$

Then, for each $d \in D_i^\tau$, $Q_i^\tau \circ Q_i^\tau(d) = Q_i^\tau(max[d]_i) = max[max[d]_i]_i = max[d]_i = Q_i^\tau(d)$. Hence Q_i^τ is a retract of D_i^τ and also of D^τ. Also, for all $d \in D_i^\tau$, $Q_i(d)$ and d belong to the same equivalence class. Hence

$$\text{for all } d \in D_i^\tau,\ Q_i(d) \simeq d.$$

Let us define

$$M_i^Q = (Q_i^\tau, \cdot, A_i^Q),$$

where . is just the restriction of the application function in M_i (or equivalently M), and $A_i^Q : \mathcal{L}_i \to \bigcup Q_i^\tau$ is defined as (dropping the type superscripts):

$$A_i^Q = Q_i \circ A_i.$$

Obviously,

$$A_i \sqsubseteq A_i^Q,$$

and

$$A_i^Q(t) \simeq A_i(t) \text{ for all } t \in \mathcal{L}_i.$$

It will turn out that M_i^Q is a fully abstract, extensional $\mathcal{L}_i$-model. Of course, a lot of work has to be done in order to prove this.

First we ask: is the application in M_i^Q well defined? That is, if $d \in Q_i^{\alpha \to \beta}$ and $c \in Q_i^\alpha$ then does $dc \in Q_i^\beta$ always? Before we address this question let us prove a general lemma.

Lemma 4.1: Suppose we are given $d_1, d_2 : \tau = \alpha \to \beta$. Then if, for all $c : \alpha$, there exists a $c' \underset{\sim}{\sqsubset} c$ such that $d_1 c \underset{\sim}{\sqsubset} d_2 c'$ then $d_1 \underset{\sim}{\sqsubset} d_2$.

Proof: We have to show that given any $(d_2, t) \in \Theta^\tau$, $(d_1, t) \in \Theta^\tau$.

Let (d_2, t) be some arbitrary element in Θ^τ. For any $(c, s) \in \Theta^\alpha$ we know that $(c', s) \in \Theta^\alpha$ as $c' \underset{\sim}{\sqsubset} c$. Hence, because $(d_2, t) \in \Theta^\tau$, $(d_2 c', ts) \in \Theta^\beta$. This implies, as $d_1 c \underset{\sim}{\sqsubset} d_2 c'$, that $(d_1 c, ts) \in \Theta^\beta$. Thus for every $(c, s) \in \Theta^\alpha$ we have $(d_1 c, ts) \in \Theta^\beta$, which means $(d_1, t) \in \Theta^\tau$. This concludes the proof.

$\diamond$

Corollary 4.2: Let $d_1, d_2 : \tau = \alpha \to \beta$ be given such that

1. $d_1 = (c \Rightarrow b)$ for some finite $c : \alpha$ and $b : \beta$, where $\Rightarrow$ indicates the usual step function, i.e.
$$\begin{aligned} d_1 a &= b \quad \text{if } c \sqsubseteq a \\ &= \bot \quad \text{otherwise.} \end{aligned}$$

2. there exists $c' \underset{\sim}{\sqsubseteq} c$ such that $b \underset{\sim}{\sqsubseteq} d_2 c'$.

Then $d_1 \underset{\sim}{\sqsubseteq} d_2$.

Proof: For every $a : \alpha$, we show that there exists $a' \underset{\sim}{\sqsubseteq} a$ such that $d_1 a \underset{\sim}{\sqsubseteq} d_2 a'$. The result then follows from the above lemma. Consider two cases.

1. $c \sqsubseteq a$: Then $c \underset{\sim}{\sqsubseteq} a$, as $\sqsubseteq$ is a refinement of $\underset{\sim}{\sqsubseteq}$. Now $d_1 a = b \underset{\sim}{\sqsubseteq} d_2 c'$ and $c' \underset{\sim}{\sqsubseteq} c \underset{\sim}{\sqsubseteq} a$, hence we can let $a' = c'$.
2. $c \not\sqsubseteq a$: Then $d_1 a = \bot$. Hence we can let $a' = \bot$.

$\diamond$

Now we can show that application in M_i^Q is well defined. Let $d \in Q_i^\tau$, where $\tau = \alpha \to \beta$, and $c \in Q_i^\alpha$. We want to show that $dc \in Q_i^\beta$. Let $b = max[dc]_i$, then this amounts to showing that $b = dc$. Define $a : D_i^\tau$ as

$$a = (c \Rightarrow b)$$

As $b = max[dc]_i$, trivially $b \underset{\sim}{\sqsubseteq} dc$. Now from the preceding corollary it immediately follows that $a \underset{\sim}{\sqsubseteq} d$. Hence $a \sqsubseteq max[a]_i \sqsubseteq max[d]_i = d$. This implies that $b = ac \sqsubseteq dc$. On the other hand $dc \sqsubseteq max[dc]_i = b$. Thus $b = dc$ and we have shown that the application in M_i^Q is well defined.

What can we say about the extensionality of M_i^Q? Note that this does *not* follow from the extensionality of M_i. The extensionality of M_i says: if $d_1, d_2 \in D_i^\tau$, where $\tau = \alpha \to \beta$, then $d_1 \sqsubseteq d_2$ whenever

$$d_1 c \sqsubseteq d_2 c \text{ for all } c : D_i^\alpha.$$

On the other hand the extensionality of M_i^Q of says: if $d_1, d_2 \in Q_i^\tau$ then $d_1 \sqsubseteq d_2$ whenever

$$d_1 c \sqsubseteq d_2 c \text{ for all } c : Q_i^\alpha,$$

which is a much stronger statement as $|Q_i^\alpha|$ is just a *sub*set of D_i^α.

We can prove extensionality as follows. Let $d_1, d_2 : Q_i^\tau$ be such that $d_1 c \sqsubseteq d_2 c$ for all $c : Q_i^\alpha$. Then for all $a : D^\alpha$ we have

$$\begin{aligned} d_1 a &= d_1(\phi_i^\alpha(a)) \quad \text{as } d_1 \in D_i^\tau \\ &\sqsubseteq d_1(Q_i^\alpha(a)) \\ &\sqsubseteq d_2(Q_i^\alpha(a)) \quad \text{by the assumption, as } Q_i^\alpha(a) \in Q_i^\alpha. \end{aligned}$$

Remembering that $\sqsubseteq$ is a refinement of $\underset{\sim}{\sqsubseteq}$, this implies $d_1 a \underset{\sim}{\sqsubseteq} d_2(Q_i^\alpha a)$ for all $a : D^\alpha$. Also, as $a \in D_i^\tau$, $Q_i^\alpha a \underset{\sim}{\sqsubseteq} a$. Now we immediately conclude from Lemma 4.1 that $d_1 \underset{\sim}{\sqsubseteq} d_2$. Hence

$$d_1 = max[d_1]_i \sqsubseteq max[d_2]_i = d_2,$$

which proves the extensionality of M_i^Q.

Algebraicity of M_i^Q follows trivially because $|Q_i^\tau|$ is finite for all τ.

Before we turn to the full abstractness of M_i^Q let us prove some lemmas.

Lemma 4.3: For all $t : \tau$, $(At, t) \in \Theta^\tau$.

Proof:

1. If t is a basic ground constant or a first order function constant then it is obvious.
2. $t = S$: We want to show that $(AS, S) = (\lambda x, y, z.(xz)(yz), S) \in \Theta$. For this it suffices to show that for all $(x, e), (y, f), (z, g) \in \Theta$, where x, y, z have appropriate *lower* order types, $((AS)xyz, Sefg) \in \Theta$.

   But $(x, e), (y, f), (z, g) \in \Theta$ implies $(xz, eg), (yz, fg) \in \Theta$ and hence

   $$((AS)xyz, (eg)(fg)) = ((xz)(yz), (eg)(fg)) \in \Theta.$$

   As $Sefg \overset{\mathcal{L}}{\simeq} (eg)(fg)$, this means

   $$((AS)xyz, Sefg) \in \Theta,$$

   which is what we wanted to prove.
3. $t = K$: The proof is as in the previous case.
4. $t = Y$: Let the type of Y be $\tau = \sigma \rightarrow \alpha$, where $\sigma = \alpha \rightarrow \alpha$. We show by induction that

   $$\text{for all } n,\ (y_n, Y) \in \Theta^\tau \text{ where } y_n = \lambda f.f^n\bot,$$

   Then the result follows from the directed completeness of Θ^τ.

   The basis is clear as $(y_0, Y) = (\bot, Y) \in \Theta^\tau$.

   Assume, as the induction hypothesis that $(y_n, Y) \in \Theta^\tau$. We have to show that $(y_{n+1}, Y) \in \Theta^\tau$. For this it suffices to show that for all $(c, s) \in \Theta^\sigma$, $(y_{n+1}c, Ys) \in \Theta^\alpha$. Let (c, s) be an arbitary element of Θ^σ. As $(y_n, Y) \in \Theta^\tau$, $(c^n\bot, Ys) = (y_n c, Ys) \in \Theta^\alpha$. Hence, as $(c, s) \in \Theta^\sigma$, $(c_{n+1}\bot, s(Ys)) = (c(c^n\bot), s(Ys)) \in \Theta^\alpha$. But, as $s(Ys) \overset{\mathcal{L}}{\simeq} Ys$, this means that $(c^{n+1}\bot, Ys) \in \Theta^\alpha$.

   Thus, for all $(c, s) \in \Theta^\sigma$, $(y_{n+1}c, Ys) = (c^{n+1}\bot, Ys) \in \Theta^\alpha$. Hence $(y_{n+1}, Y) \in \Theta^\tau$. This concludes the proof of this case.
5. $t : \beta = rs$, where $r : \alpha \rightarrow \beta$, and $s : \alpha$: By the induction hypothesis $(Ar, r) \in \Theta^{\alpha\rightarrow\beta}$ and $(As, s) \in \Theta^\alpha$. Hence $(At, t) = ((Ar)(As), rs) \in \Theta^\beta$.

◇

Corollary 4.4: For all $t : \tau \in \mathcal{L}_i$, $(A_i^Q(t), t) \in \Theta^\tau$.

Proof: Let $t \in \mathcal{L}_i$. By the above lemma, $(A_i t, t) = (At, t) \in \Theta^\tau$. Since $A_i^Q(t) \simeq A_i(t)$, this means

$$(A_i^Q(t), t) \in \Theta^\tau.$$

◇

Lemma 4.5: For $t, s : \tau$, $t \overset{\mathcal{L}}{\underset{\sim}{\sqsubseteq}} s$ iff $(At, s) \in \Theta^\tau$.

Proof:

$\Rightarrow$: Suppose $t \overset{\mathcal{L}}{\underset{\sim}{\sqsubseteq}} s$. By Lemma 4.3 $(At, t) \in \Theta^\tau$. Hence, as $t \overset{\mathcal{L}}{\underset{\sim}{\sqsubseteq}} s$, $(At, s) \in \Theta^\tau$.

$\Leftarrow$: Suppose $(At, s) \in \Theta^\tau$. Let $\tau = (\sigma_1, \ldots, \sigma_n, \kappa)$. Then for all $t_j : \sigma_j$

$$\begin{aligned} O(tt_1 \ldots t_n) &= A(tt_1 \ldots t_n) && \text{by semantic equivalence of } O \text{ and } A. \\ &= (At)(At_1) \ldots (At_n) && \\ &\sqsubseteq O(st_1 \ldots t_n) && \text{since } (At, s) \in \Theta^\tau \text{ by the assumption, and } (At_j, t_j) \in \Theta^{\sigma_j} \text{ by Lemma 4.3.} \end{aligned}$$

Thus $t \overset{\mathcal{L}}{\underset{\sim}{\sqsubseteq}} s$.

◇

Lemma 4.6: $(d, t) \in \Theta^\tau$ iff $d \underset{\sim}{\sqsubseteq} At$.

Proof:

$\Leftarrow$: Suppose $d \underset{\sim}{\sqsubseteq} At$. By Lemma 4.3 $(At, t) \in \Theta^\tau$. Hence, as $d \underset{\sim}{\sqsubseteq} At$, $(d, t) \in \Theta^\tau$.

$\Rightarrow$: Suppose $(d, t) \in \Theta^\tau$. Then for any $(At, s) \in \Theta^\tau$ we conclude from the preceding lemma that $t \overset{\mathcal{L}}{\underset{\sim}{\sqsubseteq}} s$ and hence, as $(d, t) \in \Theta^\tau$, $(d, s) \in \Theta^\tau$. This means $d \underset{\sim}{\sqsubseteq} At$.

◇

Corollary 4.7: If $d \in D_i^\tau$ and $t \in \mathcal{L}_i$ then $(d, t) \in \Theta^\tau$ implies $Q_i d \sqsubseteq A_i^Q(t)$.

Proof: By the above lemma $d \underset{\sim}{\sqsubseteq} At = A_i(t)$. This means

$$Q_i(d) = max[d]_i \sqsubseteq max[A_i(t)]_i = Q_i(A_i(t)) = A_i^Q(t).$$

◇

Corollary 4.8: $t \overset{\mathcal{L}}{\underset{\sim}{\sqsubseteq}} s$ iff $At \underset{\sim}{\sqsubseteq} As$.

Proof:

$$\begin{aligned} t \overset{\mathcal{L}}{\underset{\sim}{\sqsubseteq}} s \quad &\text{iff} \quad (At, s) \in \Theta && \text{by Lemma 4.5} \\ &\text{iff} \quad At \underset{\sim}{\sqsubseteq} As && \text{by Lemma 4.6.} \end{aligned}$$

◇

Now we are in a position to prove the full abstractness of M_i^Q. Let $t, s \in \mathcal{L}_i$ be such that $t \overset{\mathcal{L}_i}{\underset{\sim}{\sqsubset}} s$. Then by Lemma 3.1 we know that $t \overset{\mathcal{L}}{\underset{\sim}{\sqsubset}} s$. Hence by the Corollary 4.8 $A_i(t) = At \underset{\sim}{\sqsubset} As = A_i(s)$. This means

$$A_i^Q(t) = Q_i \circ A_i(t) = max[A_i(t)]_i \sqsubseteq max[A_i(s)]_i = Q_i \circ A_i(s) = A_i^Q(s).$$

Thus indeed $A_i^Q(t) \sqsubseteq A_i^Q(s)$.

On the other hand if $A_i^Q(t) \sqsubseteq A_i^Q(s)$, where $t, s \in \mathcal{L}_i$, then

$$A(t) = A_i(t) \simeq A_i^Q(t) \sqsubseteq A_i^Q(s) \simeq A_i(s) = A(s).$$

Then by Corollary 4.8, $t \overset{\mathcal{L}}{\underset{\sim}{\sqsubset}} s$. This in turn implies, by Lemma 3.1, $\lfloor t \rfloor_i \overset{\mathcal{L}_i}{\underset{\sim}{\sqsubset}} \lfloor s \rfloor_i$. But as $t, s \in \mathcal{L}_i$, we know that $t \overset{\mathcal{L}_i}{\simeq} \lfloor t \rfloor_i$ and $s \overset{\mathcal{L}_i}{\simeq} \lfloor s \rfloor_i$. Thus indeed $t \overset{\mathcal{L}_i}{\underset{\sim}{\sqsubset}} s$.

We have then proved that

$$\text{for all } t, s \in \mathcal{L}_i, t \overset{\mathcal{L}_i}{\underset{\sim}{\sqsubset}} s \text{ iff } A_i^Q(t) \sqsubseteq A_i^Q(s).$$

That is:

M_i^Q is fully abstract.

Finally we need to show that A_i^Q is a homomorphism, i.e. $A_i^Q(rs) = (A_i^Q r)(A_i^Q s)$, for all $r, s \in \mathcal{L}_i$.

Let $r, s \in \mathcal{L}_i$.

Then

$$\begin{array}{rcll} A_i^Q(rs) & = & Q_i(A_i(rs)) & \\ & = & Q_i((A_i r)(A_i s)) & \\ & \sqsubseteq & Q_i((A_i^Q r)(A_i^Q s)) & \text{as } A_i \sqsubseteq A_i^Q. \end{array} \tag{5}$$

But note that, as $A_i^Q(r), A_i^Q(s) \in Q_i$, we conclude, because the application in M_i^Q is well defined, that $(A_i^Q r)(A_i^Q s) \in Q_i$. This means $Q_i((A_i^Q r)(A_i^Q s)) = (A_i^Q r)(A_i^Q s)$. Thus by (5) we conclude that

$$A_i^Q(rs) \sqsubseteq (A_i^Q r)(A_i^Q s).$$

To prove the other inequality, it suffices to prove that $((A_i^Q r)(A_i^Q s), rs) \in \Theta$. Because then,

$$\begin{array}{rcll} (A_i^Q r)(A_i^Q s) & = & Q_i((A_i^Q r)(A_i^Q s)) & \text{as above, by the well-definedness of} \\ & & & \text{application in } M_i^Q \\ & \sqsubseteq & A_i^Q(rs) & \text{by Corollary 4.7.} \end{array}$$

It is easy to prove that $((A_i^Q r)(A_i^Q s), rs) \in \Theta$. By Corollary 4.4, $(A_i^Q(r), r), (A_i^Q(s), s) \in \Theta$. Hence $((A_i^Q r)(A_i^Q s), rs) \in \Theta$.

We have proved that

A_i^Q is a homomorphism.

Let us summarize what we have proved in this section.

M_i^Q is a fully abstract, extensional, algebraic $\mathcal{L}_i$-model. Further Q_i's are homomorphic retractions; i.e., the the following diagram commutes.

$$\begin{tikzcd}
\mathcal{L}_i \arrow[r, "A_i"] \arrow[dr, "A_i^Q"'] & M_i \arrow[d, "Q_i"] \\
& M_i^Q
\end{tikzcd}$$

Of course it is obvious that the above diagram commutes because that is how we defined A_i^Q! What is *not* obvious is that A_i^Q defined in this fashion is actually a homomorphism. It is possible to take the other approach; we can *define* A_i^Q as a homomorphism and then prove that the above diagram commutes. Which approach one takes is a matter of taste but the end result is the same anyway.

It is also interesting to note that not all the finite elements of M_i^Q at higher types need be definable, which is one of the manifestations of its truly semantic nature. We shall have more to say about this later.

5. Limit Construction

Now that we have fully abstract, extensional, algebraic finite models M_i^Q the next natural thing to do is to construct M^Q as their 'limit'. For this to go through the M_i^Qs must bear some relationship to each other. We want that M_i^Q be in some sense a *subretract* of M_j^Q if $i \leq j$.

Before that let us digress for a moment. What does it mean to say that a retract r is a subretract of a retract s? It is fair enough to stipulate that $r \sqsubseteq s$. If r and s are projections then this is all that we need, because we can then immediately conclude that $|r| \subseteq |s|$, where $|r|$ and $|s|$ are the fixpoint sets of r and s. But what if they aren't? What we need is that there be some injection-projection pair between $|r|$ and $|s|$. The obvious choice for such an injection-projection pair is the most natural one: (s, r). Pictorially:

$$\begin{tikzcd} {|r|} \arrow[r, shift left, "s"] & {|s|} \arrow[l, shift left, "r"] \end{tikzcd}$$

For (s, r) to be an injection-projection pair two conditions need to be satisfied:

1. for all $c \in s$, $s \circ r(c) \sqsubseteq c$
2. for all $d \in r$, $r \circ s(d) = d$

The first condition easily follows from the condition $r \sqsubseteq s$. The second condition is equivalent to saying $r \circ s \circ r = r$.

Hence we say $r \prec s$ if

1. $r \sqsubseteq s$
2. $r \circ s \circ r = r$.

Note that the relation $\prec$ is transitive. Also, given a monotone sequence $r_1 \prec r_2 \prec \ldots$, we can construct it's 'inverse limit' $r = \bigsqcup_{j=0}^{\infty} r_j$; this r is the least retract such that $r_j \prec r$ for all j. Also if every r_j is algebraic (by this we mean $|r_j|$ is algebraic) so is r. If d is a finite element of r_j (i.e. if d is a finite element of $|r|$; d need *not* be a finite element of the embedding domain of which r_j is a retract), then $r(d)$ is a finite element of r. Conversely every finite element of r is of this form for some j. We leave the routine proofs to the reader.

With this background we now wish to show that $Q_i^r \prec Q_j^r$, if $i \leq j$.

It is easy to see that $Q_i^r \sqsubseteq Q_j^r$. To show that $Q_i^r \circ Q_j^r \circ Q_i^r = Q_i^r$, it suffices to show that $Q_i^r \circ Q_j^r(d) = d$, for all $d \in Q_i^r$.

Consider then an arbitrary $d \in Q_i^r$. This implies $d \in D_i^r$ and hence $d \in D_j^r$ too. Now we calculate:

$$\begin{aligned} Q_i^r \circ Q_j^r(d) &= \phi_i^r(max[Q_j^r(d)]) \\ &= \phi_i^r(max[d]) && \text{because, as } d \in D_j^r,\ [Q_j^r d] = [d] \\ &= Q_i^r(d) \\ &= d && \text{as } d \in Q_i^r. \end{aligned}$$

Thus indeed

$$Q_i^\tau \prec Q_j^\tau \text{ if } i \leq j.$$

We now have, for each τ, a monotone sequence:

$$Q_1^\tau \prec Q_2^\tau \ldots Q_i^\tau \prec Q_{i+1}^\tau \prec \ldots$$

Define Q^τ as the limit of this sequence:

$$Q^\tau = \bigsqcup_{i=0}^{\infty} Q_i^\tau.$$

As $Q_i^\tau(c) \simeq c$ for every $c \in D_i^\tau$, we conclude, from the directed completeness of Θ predicates:

$$Q^\tau(d) \simeq d \text{ for every } d \in D^\tau.$$

Define $M^Q = (Q^\tau, \cdot, A^Q)$, where $\cdot$ is the restriction of the application in M and $A^Q = Q \circ A$ (dropping the type superscripts). It will turn out that M^Q is a fully abstract, extensional, algebraic $\mathcal{L}$-model.

As before we have to show that the application is well defined in M^Q, i.e., if $d \in Q^\sigma \alpha \to \beta$ and $c \in Q^\alpha$ then $dc \in Q^\beta$. But this time it is easy:

$$\begin{aligned} dc &= (\bigsqcup_{i=0}^{\infty} Q^{\sigma_{\alpha\to\beta}} i d)(\bigsqcup_{i=0}^{\infty} Q_i^\alpha c) \\ &= \bigsqcup_{i=0}^{\infty} (Q^{\sigma_{\alpha\to\beta}} i d)(Q_i^\alpha c). \end{aligned}$$

As application is well defined in each M_i^Q, we know that each $(Q^{\sigma_{\alpha\to\beta}} i d)(Q_i^\alpha c) \in Q_i^\beta$. Hence

$$dc = \bigsqcup_{i=0}^{\infty} (Q^{\sigma_{\alpha\to\beta}} i)(Q_i^\alpha) \in \bigsqcup_{i=0}^{\infty} Q_i^\beta = Q^\beta.$$

Before we prove the extensionality of M^Q let us prove one lemma.

Lemma 5.1: Let $d \in Q_j^\tau$ and $c \in Q_i^\alpha$, where $\tau = \alpha \to \beta$ and $i \sqsubseteq j$. Then

$$Q_i(d(Q_j c)) = (Q_i d)c.$$

Proof: One part of the equality, $(Q_i d)c \sqsubseteq Q_i(d(Q_i c))$ is obvious as $d \sqsubseteq Q_i(d)$ and $c \sqsubseteq Q_j(c)$. It remains to prove that $Q_i(d(Q_c)) \sqsubseteq (Q_i d)c$.

As $c \in Q_i^\alpha$, we know that $c \in D_i^\alpha$ and hence $c \in D_j^\alpha$ too. This means $Q_j(c) \simeq c$. Let $b = Q_i(d(Q_j c))$. Then

$$\begin{aligned} b &= Q_i(d(Q_j c)) \\ &\sqsubseteq Q_j(d(Q_j c)) && \text{as } Q_i \sqsubseteq Q_j \\ &= d(Q_j c) && \text{as the application in } M_i^Q \text{ is well defined.} \end{aligned}$$

Hence, as $\sqsubseteq$ is a refinement of $\underset{\sim}{\sqsubset}$, we conclude that $b \underset{\sim}{\sqsubset} d(Q_jc)$. Let $d' : D_i^\tau = (c \Rightarrow b)$. Then because $c \simeq Q_jc$ and $b \underset{\sim}{\sqsubset} (Q_jc)$, we immediately conclude from Lemma 4.1 that $d' \underset{\sim}{\sqsubset} d$. Therefore $d' \sqsubseteq d$, as $d' \sqsubseteq max[d']_j \sqsubseteq max[d]_j = d$. But then, as $d' \in D_i^\tau$, $d' \sqsubseteq Q_i(d') \sqsubseteq Q_i(d)$. Hence $Q_i(d(Q_ic)) = b = d'c \sqsubseteq (Q_id)c$.

◇

Lemma 5.1 reminds us of the following fact in the model M which can be easily proved from the definitions of the ϕ_i projections.

If $d \in D_j^\tau$ and $c \in D_i^\alpha$, where $\tau = \alpha \to \beta$ and $i \leq j$ then $\phi_i(d(\phi_jc)) = (\phi_id)c$.

Of course, as $c \in D_i^\alpha$ implies $\phi_j(c) = c$, the above equation can be reduced to: $\phi_i(dc) = (\phi_id)c$, whereas in M^Q we can not conclude more than the fact $c \sqsubseteq Q_j(c)$.

From the above Lemma 5.1 one easily proves that the following diagram commutes:

$$\begin{tikzcd}
Q_j^{\alpha\to\beta} \times Q_j^\alpha \arrow[r, "\cdot"] & Q_j^\beta \arrow[d, "Q_i^\beta"] \\
Q_i^{\alpha\to\beta} \times Q_i^\alpha \arrow[u, "{(Q_j^{\alpha\to\beta}, Q_j^\alpha)}"] \arrow[r, "\cdot"] & Q_i^\beta
\end{tikzcd}$$

This roughly says that the application remains invariant under the injection of the family $\{Q_i^\tau\}$ into $\{Q_j^\tau\}$ which is slightly surprising, as this injection 'increases' the elements: $Q_j^\tau(c) \sqsupseteq c$, for $c : Q_i^\tau$. Thus in true sense $\{Q_i^\tau\}$ can be *embedded* into $\{Q_j^\tau\}$. We say:

$$\{Q_i^\tau\} \lhd \{Q_j^\tau\}.$$

Corollary 5.2: If $d \in Q^\tau$ and $c \in Q_i^\alpha$, where $\tau = \alpha \to \beta$, then $Q_i(d(Qc)) = (Q_id)c$.

Proof:

$$\begin{aligned}
Q_i(d(Qc)) &= Q_i(\bigsqcup_{j=0}^{\infty}(Q_jd)(Q_j \circ Q(c))) \\
&= Q_i(\bigsqcup_{j=0}^{\infty}(Q_jd)(Q_j(c))) \\
&= \bigsqcup_{j\geq i}^{\infty} Q_i((Q_jd)(Q_jc)) \\
&= \bigsqcup_{j\geq i}^{\infty}(Q_id)c \qquad\qquad \text{by Lemma 5.1} \\
&= (Q_id)c.
\end{aligned}$$

◇

I am sure Corollary 5.2 reminds the reader of the following fact in the model M which could proved analogusly:

If $d \in D^{\tau}$ and $c \in D_i^{\alpha}$, where $\tau = \alpha \to \beta$, then

$$\phi_i(d(\phi c)) = (\phi_i d)c$$

Proving extensionality now is easy. Suppose $a, b \in Q^{\tau}$, where $\tau = \alpha \to \beta$, and that for all $c \in Q^{\alpha}$, $ac \sqsubseteq bc$. Then for all $h \in Q_i^{\alpha}$,

$$\begin{array}{rcll} (Q_i^{\tau}a)h & = & Q_i^{\beta}(a(Q^{\alpha}h)) & \text{by Corollary 5.2} \\ & \sqsubseteq & Q_i^{\beta}(b(Q^{\alpha}h)) & \text{by assumption, as } Q^{\alpha}(h) \in Q^{\alpha} \\ & = & (Q_i^{\tau}b)h & \text{by Corollary 5.2.} \end{array}$$

By extensionality of M_i^Q, we conclude that $Q_i^{\tau}(a) \sqsubseteq Q_i^{\tau}(b)$. Hence

$$a = Q^{\tau}(a) = \bigsqcup_{i=0}^{\infty} Q_i^{\tau}(a) \sqsubseteq \bigsqcup_{i=0}^{\infty} Q_i^{\tau}(b) = Q^{\tau}(b) = b,$$

which proves that M^Q *is extensional.*

Note that for every $t \in \mathcal{L}$,

$$\begin{array}{rcl} A^Q(t) & = & Q(At) \\ & = & (\bigsqcup_{i=0}^{\infty} Q_i)(\bigsqcup_{i=0}^{\infty} A_i(\lfloor t \rfloor_i)) \\ & = & \bigsqcup_{i=0}^{\infty} Q_i \circ A_i(\lfloor t \rfloor_i) \\ & = & \bigsqcup_{i=0}^{\infty} A_i^Q(\lfloor t \rfloor_i). \end{array}$$

Also $A \sqsubseteq A^Q$, because for every $t \in \mathcal{L}$

$$\begin{array}{rcll} A(t) & = & \bigsqcup_{i=0}^{\infty} A_i(\lfloor t \rfloor_i) & \\ & \sqsubseteq & \bigsqcup_{i=0}^{\infty} A_i^Q(\lfloor t \rfloor_i) & \text{as } A_i \sqsubseteq A_i^Q \\ & = & A^Q(t). & \end{array}$$

A^Q is a homomorphism,

because for every $r, s \in \mathcal{L}$ of appropriate type,

$$\begin{aligned}
A^Q(rs) &= Q \circ A(rs) \\
&= (\bigsqcup_{i=0}^{\infty} Q_i)(\bigsqcup_{i=0}^{\infty} A(\lfloor r\rfloor_i \lfloor s\rfloor_i)) && \text{by syntactic continuity} \\
&= (\bigsqcup_{i=0}^{\infty} Q_i)(\bigsqcup_{i=0}^{\infty} A_i(\lfloor r\rfloor_i \lfloor s\rfloor_i)) \\
&= \bigsqcup_{i=0}^{\infty} Q_i \circ A_i(\lfloor r\rfloor_i \lfloor s\rfloor_i) \\
&= \bigsqcup_{i=0}^{\infty} A_i^Q(\lfloor r\rfloor_i \lfloor s\rfloor_i) \\
&= \bigsqcup_{i=0}^{\infty} (A_i^Q \lfloor r\rfloor_i)(A_i^Q \lfloor s\rfloor_i) && \text{because } A_i^Q \text{ is a homomorphism} \\
&= (\bigsqcup_{i=0}^{\infty} A_i^Q(\lfloor r\rfloor_i))(\bigsqcup_{i=0}^{\infty} A_i^Q(\lfloor s\rfloor_i)) \\
&= (A^Q r)(A^Q s).
\end{aligned}$$

M^Q is algebraic because it is the inverse limit of M_i^Qs and each M_i^Q is trivially algebraic.

We now address the question of full abstractness of M^Q.

Suppose $t \overset{\mathcal{L}}{\underset{\sim}{\sqsubset}} s$. Then by Lemma 3.1, we know that for all i, $\lfloor t\rfloor_i \overset{\mathcal{L}_i}{\underset{\sim}{\sqsubset}} \lfloor s\rfloor_i$. This is because full abstractness of M_i^Q implies $A_i^Q(\lfloor t\rfloor_i) \sqsubseteq A_i^Q(\lfloor s\rfloor_i)$, for all i. Hence

$$A^Q t = \bigsqcup_{i=0}^{\infty} A_i^Q(\lfloor t\rfloor_i) \sqsubseteq \bigsqcup_{i=0}^{\infty}(\lfloor s\rfloor_i) = A^Q(s).$$

On the other hand suppose $A^Q(t) \sqsubseteq A^Q(s)$. Then:

$$A(t) \sqsubseteq A^Q(t) \sqsubseteq A^Q(s) = Q(As) \simeq A(s).$$

Thus $A(T) \underset{\sim}{\sqsubset} A(S)$, which, by Corollary 4.8, means that $t \overset{\mathcal{L}}{\underset{\sim}{\sqsubset}} s$. We have proved:

M^Q is fully abstract.

Y has the standard interpretation in M^Q, i.e., $A^Q(Y) = \bigsqcup_{j=0}^{\infty} A^Q(Y_j)$, where $Y_j = \lambda f.f^n(\Omega)$.

This is because

$$\begin{aligned}
A^Q(Y) &= Q(AY) \\
&= Q(\bigsqcup_{j=0}^{\infty} A(Y_j)) \\
&= \bigsqcup_{j=0}^{\infty} Q \circ A(Y_j) \\
&= \bigsqcup_{j=0}^{\infty} A^Q(Y_j).
\end{aligned}$$

M^Q is also a β-model (i.e. a model for beta-conversion). For this one has to prove that

1. $A^Q(Suvw) = A^Q((uw)(vw))$ for all $u, v, w \in \mathcal{L}$ of appropriate types.
2. $A^Q(Kuv) = A^Q(u)$ for all $u, v \in \mathcal{L}$ of appropriate types.

These equations say that S behaves like S, K behaves like K, and they constitute a closed combinator version of the usual beta-conversion equation. But they are obviously true in M^Q because $Suvw \overset{\mathcal{L}}{\simeq} (uw)(vw)$, $Kuv \overset{\mathcal{L}}{\simeq} u$ and M^Q is fully abstrtact.

To summarize:

> *M^Q is a fully abstract, extensional, algebraic β-model for $\mathcal{L}$. Y has the standard interpretation in M^Q and moreover $A^Q = Q \circ A$.*

Now that we have proved that M^Q is the desired model, we can give its equivalent *direct* definition. Note that for each $d \in D^\tau_i$,

$$Q^\tau(d) = \bigsqcup_{j \geq i}^{\infty} Q^\tau_j(d) = \bigsqcup_{j > i}^{\infty} max[d]_j = max[d].$$

Let us define a monotonic function, F^τ, on the finite elements of D^τ:

$$F^\tau(d) = max[d], \quad \text{for each finite } d \in D^\tau. \tag{6}$$

It follows that

> *Q^τ is the unique continuous extension of F^τ.*

Similarly, as we know now that A^Q is a homomorphism, we can give its equivalent *denotational* definition:

1. $A^Q(b) = Q(b)$ where b is a basic ground type constant.
2. $A^Q(f) = Q(f)$ where f is a basic first order function constant.
3. $A^Q(S) = Q(\lambda x, y, z.(xz)(yz))$
4. $A^Q(K) = Q(\lambda x, y.x)$
5. $A^Q(Y) = Q(\bigsqcup_{n=0}^{\infty} \lambda f.f^n(\bot))$
6. $A^Q(rs) = (A^Q r)(A^Q s)$ where r and s are of the appropriate types.

What we have proved can now be *rephrased* as the following main result of this paper:

Theorem 5.3: $M^Q = (Q^\tau, \cdot, A^Q)$, where

1. Q^τ is defined as a continuous extension of F^τ in eq 6,
2. the application $\cdot$ is just the restriction of the application function of M,
3. A^Q is defined *denotationally* as above,

is a fully abstract, extensional, algebraic, β-model of $\mathcal{L}$. Moreover Y has the standard interpretation in M^Q and the following diagram commutes.

$$\begin{tikzcd}
\mathcal{L} \arrow[r, "A"] \arrow[dr, "A^Q"'] & M \arrow[d, "Q"] \\
 & M^Q
\end{tikzcd}$$

$\diamond$

Note that now we are justified in letting the application in M^Q be just the restriction of the application in M. Also note that the statement of Theorem 5.3 is direct in the sense it does not make any reference to the finite approximate models M_i^Q, although they were used in its proof. It is very pleasing that M^Q can be constructed directly in a straightforward fashion, though proving that it has all the desired properties was certainly not straightforward! In fact, on the surface it seems quite unlikely that the application will be well defined in M^Q or that it will be extensional.

We can now easily show that $t \overset{\mathcal{L}}{\underset{\sim}{\sqsubseteq}} s$ iff for all ground contexts $C[\;]$,

$$O(C[t]) \sqsubseteq O(C[s]). \tag{7}$$

Suppose $t \overset{\mathcal{L}}{\underset{\sim}{\sqsubseteq}} s$. Then, because M^Q is fully abstract, $A^Q(t) \sqsubseteq A^Q(s)$. This means, as A^Q is a homomorphism, that $A^Q(C[t]) \sqsubseteq A^Q(C[s])$ for all ground contexts $C[\;]$. But then,

$$\begin{array}{rcll} O(C[t]) & = & A^Q(C[t]) & \text{as } M^Q \text{ is adequate} \\ & \sqsubseteq & A^Q(C[s]) & \\ & = & O(C[s]) & \text{as } M^Q \text{ s adequate.} \end{array}$$

On the other hand if $O(C[t]) \sqsubseteq O(C[s])$ for all ground contexts then $O(tt_1 \dots t_n) \sqsubseteq O(st_1 \dots t_n)$ for all $t_1, \dots, t_n \in \mathcal{L}$ of appropriate types (let context be $[\;]t_1 \dots t_n$. And hence $t \overset{\mathcal{L}}{\underset{\sim}{\sqsubseteq}} s$.

Thus (7) could be used as an alternative definition of $\overset{\mathcal{L}}{\underset{\sim}{\sqsubseteq}}$ instead of the one given in (1), because both of them are now shown to be equivalent. It is surprising how naturally this equivalence follows from the *existence* of fully abstract M^Q. Contrast this with the elaborate efforts taken in [Milner] to prove this equivalence (Milner's First Context Lemma). Milner could not have taken our approach because, unlike in our case, the construction of his fully abstract model *depends upon* the validity of the above equivalence. Of course there is a hidden assumption in our argument, namely all the finite projections of the ground domains are definable, because this assumption was used in the construction of M^Q, whereas Milner's First Context Lemma does not need any assumption like this. BUT, what we have proved here is more general than Milner's First Context Lemma because the First Context Lemma is *not* applicable when the language contains Y combinator. In the presence of Y combinator an addtional assumption seems necessary anyway.

6. Discussion

Observe that in our fully abstract model M^Q not all the finite elements at the higher type need be definable. I have not tried to come up with a specific finite element which is not definable. But I conjecture this is the case. Contrast this with Milner's syntactic construction of a fully abstract model for typed lambda calculus, where every finite element is definable. Even when the classical model of continuous functions was shown fully abstract in [Plotkin] for PCF enriched with 'parallel or' it turned out that all the finite elements of the model were definable in the enriched language. To my knowledge, this is the first model which is fully abstract and whose all finite elements need not be definable still. We leave it open to find out the properties which the ground domains should have so that all the finite elements of M^Q are definable. (see articulate domains of [Milner])

We also leave open whether a *better* semantic characterization can be found when $\mathcal{L}$ is known to possess the *addtional* properties like sequentiality (see [Berry]).

Let us recollect the assumption on which Theorem 5.3 was proved.

Basic Assumption

1. The ground domains D^κ are algebraic, complete lattices.
2. All the finite elements of the ground domains as well as their finite projections are definable.

That we are using complete lattices instead of consistently complete cpos is a technical problem which we would neglect at the moment. Otherwise our assumption is *exactly* the same as the assumption Milner used to construct a *syntactic* fully abstract, extensional model for $\mathcal{L}$. This surprising coincidence makes us speculate that the *Basic Assuption* might in some sense be a necessary condition to guarantee the *existence* of a fully abstract, extensional model for $\mathcal{L}$. We can justify it as follows. It seems reasonable to assume that if some fully abstract, extensional model for $\mathcal{L}$ exists then one can construct such a model syntactically. After all the syntactic construction is the easiest one, and the whole point of Milner's syntactic construction in [Milner] was to show that a fully abstract, extensional model for $\mathcal{L}$ exists. If one can show that the *Basic Assumption* is in some sense necessary for Milner's syntactic construction to succeed, then it follows that the *Basic Assumption* is in some sense a necessary condition for the existence of a fully abstract, extensional model for $\mathcal{L}$. It is then not surprising that we hit upon the same condition in the semantic characterization of full abstraction; you don't expect to find such a characterization when a fully abstract model does not exist!

What is important: this paper gives us a definite step in the opposite direction; whenever the *Basic Assumption* is true, a semantic characterization can be found. Then we can say:

> *If there exists some fully abstract, extensional, algebraic model for $\mathcal{L}$ then such a model can be found semantically.*

It is a rather sweeping statement, but one can not resist the temptation! I would expect something of this sort, with possibly a variant of the *Basic Assumption*, to be true. That we leave as an open question for the moment.

Though we did not show it, the Θ predicates defined in this paper can be used to show that O and A are semantically equivalent. In fact such inclusive predicates were introduced in [Milne] and [Reynolds] with exactly this aim in mind: to show the semantic equivalence between operational and denotational semantics. The techniques developed in these papers were mainly meant for the cases when the domains under consideration were reflexive. It should not then come as a surprise if the technique developed in this paper could be extended to obtain a semantic characterization of full abstraction even when domains under consideration are reflexive. In fact that *is* the case.

The same inclusive predicates which are used to show that the operational and denotational semantics are equivalent can used to collapse the model onto a fully abstract one.

This ties everything together nicely. In the rest of the paper we shall extend the theory to take into account the presence of reflexive domains.

7. Reflexive Types

We extend our language by adding reflexive types. Assume without loss of generality that Γ contains a 'bottom' type ω such that D^{ω} is a trivial one point domain. (For example, one can assume that ω was one of the ground types.) We grant ω a special status henceforth. We extend Γ by throwing in the reflexive types. Let $\bar{\Gamma}$, the set of reflexive types, be the least set such that:

1. $\omega \in \bar{\Gamma}$, where ω is a bottom type,
2. $\kappa \in \bar{\Gamma}$, where κ is a ground type,
3. $\alpha \to \beta \in \bar{\Gamma}$, if $\alpha, \beta \in \bar{\Gamma}$,
4. $!\alpha.g(\alpha) \in \bar{\Gamma}$, if g is a type expression over $\bar{\Gamma}$ with one free variable α.

It is easy to see that $\Gamma \subseteq \bar{\Gamma}$.

Let $\approx$ be the smallest congruence relation such that:

$$\alpha \approx g(\alpha) \text{ if } \alpha = !\tau.g(\tau).$$

We shall let $\alpha, \tau, \alpha_1 \ldots$ to range over $\bar{\Gamma}$.

Define a family $\{\bar{\mathcal{L}}^{\tau}\}$ as the family consisting of the smallest sets $\bar{\mathcal{L}}^{\tau}$, where $\bar{\mathcal{L}}^{\tau}$ is intended to be a set of reflexive terms of type τ, such that:

1. $\mathcal{B}^{\kappa} \subseteq \bar{\mathcal{L}}^{\kappa}$, where κ is a ground type,
2. $\mathcal{F}^{\tau} \subseteq \bar{\mathcal{L}}^{\tau}$, where τ is a first order type,
3. $S, K, Y \in \bar{\mathcal{L}}^{\tau}$, if they are of types τ,
4. $ts \in \bar{\mathcal{L}}^{\beta}$, if $t \in \bar{\mathcal{L}}^{\alpha \to \beta}$ and $s \in \bar{\mathcal{L}}^{\alpha}$,
5. $t \in \bar{\mathcal{L}}^{\alpha}$, if $t \in \bar{\mathcal{L}}^{\beta}$ and $\alpha \approx \beta$.

It is easy to see that $\mathcal{L}^{\tau} \subseteq \bar{\mathcal{L}}^{\tau}$, if $\tau \in \Gamma$.

As usual we write $t : \tau$ to say $t \in \bar{\mathcal{L}}^{\tau}$. But now a term might possess many types: if $t : \alpha$ and $\alpha \approx \beta$ then $t : \beta$.

Next we extend the nonreflexive reduction relation $\longrightarrow$. We assume that we are given $\overset{f}{\rightharpoonup}$, a partial function from $\bar{\mathcal{L}}^{\kappa_1} \times \cdots \times \bar{\mathcal{L}}^{\kappa_n}$ to $\mathcal{B}^{\kappa}$, which is an extension of $\xrightarrow{f}$. We extend $\longrightarrow$ to $\rightharpoonup$ as follows.

1. $\dfrac{f t_1 \ldots t_n \overset{f}{\rightharpoonup} b}{f t_1 \ldots t_n \rightharpoonup b}$ where $t_j \in \bar{\mathcal{L}}^{\kappa_j}$, $b \in \mathcal{B}$ and $f \in \mathcal{F}$.
2. $(Y\ f) \rightharpoonup f\ (Y\ f)$ where $f \in \bar{\mathcal{L}}^{\sigma \to \sigma}$.
3. $S\ r\ s\ t \rightharpoonup (r\ t)\ (s\ t)$
4. $K\ r\ s \rightharpoonup r$
5. $\dfrac{r \rightharpoonup r'}{r\ s \rightharpoonup r'\ s}$
6. $\dfrac{s \rightharpoonup s'}{r\ s \rightharpoonup r\ s'}$

We shall denote by $\overset{*}{\rightharpoonup}$ the transitive reflexive closure of $\rightharpoonup$; it will turn out to be an extension of $\xrightarrow{*}$.

We assume that $\xrightarrow{f}$ is reasonable, i.e. it satisfies the following consistency constraint:

if $ft_1 \dots t_n \xrightarrow{f} b$ and $t_j \xrightarrow{*} s_j$, for all j, then $fs_1 \dots s_n \xrightarrow{f} b$.

The notion of a model is analogous to the one in the nonreflexive case, and hence is left to the reader.

To construct a classical model for $\bar{\mathcal{L}}$ we shall embed everything into a universal domain U. Assume that for each ground type κ we are given a finitary projection of the universal domain, ϕ^κ. Let $D^\kappa = |\phi^\kappa|$. Define ϕ^τ at the higher types by induction:

1. $\phi^\omega = \bot$,
2. ϕ^κ is as given, if κ is a ground type,
3. $\phi^\tau = \lambda f.\, \phi^\beta \circ f \circ \phi^\alpha$, if $\tau = \alpha \to \beta$,
4. $\phi^\tau = \bigsqcup_{n=0}^{\infty} \{\phi^{\tau_n} \mid \tau_n = g^n(\omega)\}$, if $\tau = !\alpha.g(\alpha)$.

We denote $|\phi^\tau|$ by D^τ. Note that if $\alpha \approx \beta$ then $\phi^\alpha = \phi^\beta$ and hence $D^\alpha = D^\beta$.

Though we did not carry out the treatment of nonreflexive types in the universal domain, we could have done so, and hence we shall assume that the treatement of the previous sections was carried out in the universal domain U.

Let $\bar{M} = (D^\tau, \cdot, \bar{A})$ be the classical model of $\bar{\mathcal{L}}$, where $\cdot$ is the usual application function and $\bar{A}$ is defined as follows:

1. $\bar{A}(b) = G(b)$, for $b \in \mathcal{B}$,
2. $\bar{A}(f) = H(f)$, for $f \in \mathcal{F}$,
3. $\bar{A}(S) = \lambda xyz.\, (xz)(yz)$,
4. $\bar{A}(K) = \lambda xy.\, x$,
5. $\bar{A}(Y) = \bigsqcup_{n=0}^{\infty} \lambda f.\, f^n(\bot)$,
6. $\bar{A}(t\, s) = (\bar{A}t)\, (\bar{A}s)$.

$\bar{A}$ is an extension of A, i.e. if $t \in \mathcal{L}$ then $\bar{A}(t) = A(t)$.

Let us define a type-respecting map $\bar{O} : \bigcup \bar{\mathcal{L}}^\kappa \to \bigcup D^\kappa$ as:

$$\bar{O}\, t = \begin{cases} G\, b & \text{if } t \xrightarrow{*} b \\ \bot & \text{otherwise, i.e. if all computation of } t \text{ diverge.} \end{cases}$$

Then $\bar{O}$ is an extension of O. We shall soon show that $\bar{M}$ is adequate, i.e. $\bar{A}(t) = \bar{O}(t)$ if $t \in \bar{\mathcal{L}}$ is a ground type term.

The notion of full abstractness is as in the nonreflexive case. We say $t \overset{\mathcal{L}}{\underset{\sim}{\sqsubseteq}} s$, where $t, s : \tau$, if

$$O(tt_1 \dots t_n) = O(st_1 \dots t_n),$$

whenever $\tau \approx (\tau_1, \dots, \tau_n, \kappa)$ and $t_j : \tau_j$, for all j. It will turn out that this definition is equivalent to the one given in terms of contexts. It differs from the one for the nonreflexive case in two ways.

Firstly τ may be congruent to *many* types of the form $(\tau_1, \ldots, \tau_n, \kappa)$. But if τ is congruent to $(\tau_1, \ldots, \tau_n, \kappa)$ and $(\tau_1', \ldots, \tau_n', \kappa')$ then $n = n'$, $\kappa = \kappa'$, and $\tau_j = \tau_j'$ which means $\bar{\mathcal{L}}^{\tau_j} = \bar{\mathcal{L}}^{\tau_j'}$. Thus it suffices to consider *any* one congruent type of the form $(\tau_1, \ldots, \tau_n, \kappa)$.

The second difference is more subtle: τ may not be congruent to any type of the form $(\tau_1, \ldots, \tau_n, \kappa)$; in this case we say τ is trivial. From the definition of $\overset{\mathcal{L}}{\underset{\sim}{\sqsubseteq}}$ it follows that for all $t, s : \tau$, $t \overset{\mathcal{L}}{\simeq} s$! But this is not surprising because it is easy to show by induction on the definion of types that if τ is trivial then $\phi^\tau = \bot$. As $\bar{M}$ is adequate (we shall soon show that), this means $\bar{O}(t) = \bar{A}(t) = \bot$, for all $t \in \bar{\mathcal{L}}^\tau$, hence all $t : \tau$ are equivalent.

Adequacy of $\bar{M}$ is shown by defining for each type an inclusive predicate $\bar{\Theta}^\tau$. (The same approach could have been used to prove adequacy of M as well.) There is nothing new in this; such proofs can be found in [Milne] and [Reynolds] for example. What *is* new is that we can use the same predicates to collapse M onto a fully abstract, extensional model. This strengthens our belief that the proofs of semantic equivalence and full abstraction should go hand in hand.

For each type τ let R^τ be the set of directed complete relations $\Sigma^\tau \subseteq D^\tau \times \bar{\mathcal{L}}^\tau$. We shall inductively construct for each type τ a predicate $\bar{\Theta}^\tau \in R^\tau$. We have to consider 4 cases.

1. $\tau = \omega$. Then
$$\bar{\Theta}^\tau = \{(\bot, t) \mid t \in \bar{\mathcal{L}}^\tau\}.$$

2. $\tau = \kappa$. Then
$$\bar{\Theta}^\tau = \{(d, t) \mid d \sqsubseteq \bar{O}(t)\}.$$

3. $\tau = \alpha \to \beta$. Then
$$\bar{\Theta}^\tau = \{(d, t) \mid \text{ for all } (c, s) \in \bar{\Theta}^\alpha.(dc, ts) \in \bar{\Theta}^\beta\}.$$

4. $\tau =!\alpha.g(\alpha)$. Let C be the set of types $\beta \in \bar{\Gamma}$ occuring in g. By induction hypothesis we assume that $\bar{\Theta}^\beta$ has already been constructed for each $\beta \in C$.

   Let e be any type expression with at most one free variable α such that every type $\beta \in \bar{\Gamma}$ occuring in e belongs to C. We inductively define $F_{e(\tau)} : R^\tau \to R^{e(\tau)}$. For every $\Sigma^\tau \in R^\tau$:

   (a) if $e(\tau) = \beta$, where $\beta \in C$, then
$$F_{e(\tau)}(\Sigma^\tau) = \bar{\Theta}^\beta,$$

   (b) if $e(\tau) = \tau$ then
$$F_{e(\tau)}(\Sigma^\tau) = \Sigma^\tau,$$

   (c) if $e(\tau) = e_1(\tau) \to e_2(\tau)$ then
$$F_{e(\tau)}(\Sigma^\tau) = \{(d, t) \mid \text{ for all } (c, s) \in F_{e_1(\tau)}(\Sigma^\tau).(dc, ts) \in F_{e_2(\tau)}(\Sigma^\tau)\}.$$

Let $\bar{\Theta}_0^\tau = \{(\bot, s) \mid s \in \bar{\mathcal{L}}^\tau\}$, and, for each n, let $\bar{\Theta}_{n+1}^\tau = F_{g(\tau)}(\bar{\Theta}_n^\tau)$, and $\tau_n = g^n(\omega)$. Note that each $\bar{\Theta}_n^\tau \subseteq D^{\tau_n} \times \bar{\mathcal{L}}^\tau$. Finally, let

$$\bar{\Theta}^\tau = \{(d, s) \mid \text{ for all } n.(\phi^{\tau_n}(d), s) \in \bar{\Theta}_n^\tau\}.$$

Lemma 7.1: If $\tau =! \alpha.g(\alpha)$ then $\bar{\Theta}^\tau = \bar{\Theta}^{g(\tau)}$. (Note that, as $\phi^\tau = \phi^{g(\tau)}$ and $\bar{\mathcal{L}}^\tau = \bar{\mathcal{L}}^{g(\tau)}$, both $\bar{\Theta}^\tau$ and $\bar{\Theta}^{g(\tau)}$ are the subsets of $D^\tau \times \bar{\mathcal{L}}^\tau$.)

Proof: The proof is similar to the ones in [Milne], [Reynolds] or [Mulmuley], hence we omit it.

◇

Corollary 7.2: If $\alpha \approx \beta$ then $\bar{\Theta}^\alpha = \bar{\Theta}^\beta$

◇

The above corollary yields us a very convenient equivalent definition of the $\bar{\Theta}^\tau$ predicates. If τ is trivial then it is easy to show that

$$\bar{\Theta}^\tau = \{(\bot, t) \mid t \in \bar{\mathcal{L}}^\tau\}.$$

Otherwise let $\tau \approx (\tau_1, \ldots, \tau_n, \kappa)$. Then $\bar{\Theta}^\tau = \bar{\Theta}^{(\tau_1, \ldots, \tau_n, \kappa)}$. Hence

$$\bar{\Theta}^\tau = \{(d, t) \mid \text{ for all } (c_i, s_i).\, dc_1 \ldots c_n \sqsubseteq \bar{O}(ts_1 \ldots s_n)\}.$$

Lemma 7.3: $(\bar{A}(t), t) \in \bar{\Theta}^\tau$ for all $t \in \bar{\mathcal{L}}^\tau$.

Proof: The proof is very similar to that of Lemma 4.3, except for one new case: $t \in \bar{\mathcal{L}}^\alpha$ and $\tau \approx \alpha$. But in this case, by Corollary 7.2, $\bar{\Theta}^\tau = \bar{\Theta}^\alpha$.

◇

Now adequacy of $\bar{M}$ is easy to prove. Let $t : \kappa$ be any ground term. Then by the above lemma $(\bar{A}(t), t) \in \bar{\Theta}^\kappa$. Hence, from the definition of $\bar{\Theta}^\kappa$, $\bar{A}(t) \sqsubseteq O(t)$.

On the other hand, it is easy to show that if $t \rightarrow s$ then $\bar{A}(t) = \bar{A}(s)$. Hence if $t \xrightarrow{*} b$ then $\bar{A}(t) = b$, which means $\bar{A}(t) \sqsupseteq \bar{O}(t)$.

Thus $\bar{A}(t) = \bar{O}(t)$ for all ground terms t, i.e.

$\bar{M}$ is adequate.

8. The Collapsed Model

In the this section we see how we can use the same $\bar{\Theta}$ predicates to collapse $\bar{M}$ onto a fully abstract, extensional model $\bar{M}^Q$.

Given $d_1, d_2 \in D^\tau$, we say $d_1 \bar{\underset{\sim}{\sqsubset}}^\tau d_2$ if $(d_2, t) \in \bar{\Theta}^\tau$ implies $(d_1, t) \in \bar{\Theta}^\tau$. Let $\bar{\simeq}^\tau$ be the induced equivalence relation. The equivalence class of d will be denoted by $[\bar{d}]^\tau$.

We say $[\bar{d}]_1 \bar{\underset{\sim}{\sqsubset}} [\bar{d}]_2$ if $d_1 \bar{\underset{\sim}{\sqsubset}} d_2$. (The choice of the representatives d_1 and d_2 does not matter.) Let $D_\sim$ be the quotient space induced by $\bar{\simeq}$. Then $D_\sim$ is a cpo under this ordering.

Define a monotonic function $\bar{F}^\tau$ on the finite elements of D^τ:

$$\bar{F}^\tau(d) = max[\bar{d}], \text{ for each finite } d \in D^\tau.$$

Let $\bar{Q}^\tau$ be the unique continuous extension of $\bar{F}^\tau$. Let $\bar{M}^Q = (D^\tau, \cdot, \bar{A}^Q)$, where $\cdot$ is the restriction of the application in $\bar{M}$ and $\bar{A}^Q = \bar{Q} \circ \bar{A}$. We shall show that $\bar{M}^Q$ is a fully abstract, extensional, algebraic model of $\bar{\mathcal{L}}$.

As in the nonreflexive case we show that $\bar{M}^Q$ is the limit of a set of models.

For each i, define ϕ_i^τ inductively:

1. $\phi_i^\omega = \perp$,
2. ϕ_i^κ is as given, if κ is a ground type,
3. $\phi_i^\tau = \lambda f : \tau.(\phi_i^\beta \circ f \circ \phi_i^\alpha)$, if $\tau = \alpha \rightarrow \beta$,
4. $\phi_i^\tau = \bigsqcup_{n=0}^{\infty} \phi_i^{\tau_n}$, if $\tau = !\alpha.g(\alpha)$; define $\tau_n = g^n(\omega)$.

Let $D_i^\tau = |\phi_i^\tau|$. Each ϕ_i^τ can be shown to be definable by some term $\Phi_i^\tau : \tau$. (We have already shown this when $\tau \in \Gamma$.)

Let $[\bar{d}]_i^\tau = [\bar{d}]^\tau \cap D_i^\tau$. Define a monotonic map F_i^τ on the finite elements of D_i^τ:

$$\bar{F}_i^\tau(d) = max[\bar{d}]_i^\tau, \text{ for each } d \in D_i^\tau.$$

Let $\bar{Q}_i^\tau$ be the unique extension of $\bar{F}_i^\tau$. Of course when $\tau \in \Gamma$ then $\bar{Q}_i^\tau = F_i^\tau$ and, hence, $\bar{Q}_i^\tau$ is finite. Also, for each $\tau \in \Gamma$,

$$\bar{Q}^\tau = \bigsqcup_{i=0}^{\infty} \bar{Q}_i^\tau.$$

The first natural question which comes to mind is: what is the relationship between $\bar{Q}_i^\tau$ and $\bar{Q}_i^\tau$ when $\tau \in \Gamma$?

9. Relation With The Nonreflexive Model

In this section we show that $\bar{Q}^{\tau}_{i} = Q^{\tau}_{i}$ if $\tau \in \Gamma$.

Let $\prec$ be the smallest quasiorder on $\bar{\Gamma}$ such that:

1. $\omega \prec \tau$ for all $\tau \in \bar{\Gamma}$,
2. $\alpha_1 \rightarrow \beta_1 \prec \alpha_2 \rightarrow \beta_2$ if $\alpha_1 \prec \beta_1$ and $\alpha_2 \prec \beta_2$,
3. $\prec$ is $\approx$-respecting, i.e. if $\alpha_1 \approx \beta_1$, $\alpha_2 \approx \beta_2$ and $\alpha_1 \prec \beta_1$ then $\alpha_2 \prec \beta_2$.

Note that the equivalence relation induced by $\prec$ is presicely $\approx$. Also if $\alpha_1 \rightarrow \alpha_2 \prec \beta_1 \rightarrow \beta_2$ then $\alpha_1 \prec \beta_1$ and $\alpha_2 \prec \beta_2$. It is easy to see that

$$g^n(\omega) \prec \tau \text{ if } \tau = !\alpha.g(\alpha).$$

Moreover if $\alpha \prec \beta$ then $\phi^{\alpha} \sqsubseteq \phi^{\beta}$ and hence $D^{\alpha} \subseteq D^{\beta}$. We next show that the injection-projection pair between D^{α} and D^{β}, where $\alpha \prec \beta$, is definable, i.e. there exist $\Psi^{\beta}_{\alpha} : \alpha \rightarrow \beta$, $\Phi^{\beta}_{\alpha} : \beta \rightarrow \alpha$ such that:

1. $\bar{A}(\Phi^{\beta}_{\alpha})d = \phi^{\alpha}(d)$ for all $d \in D^{\beta}$,
2. $\bar{A}(\Psi^{\beta}_{\alpha})d = d$ for all $d \in D^{\alpha}$.

In fact we shall show that:

1. $\bar{A}(\Phi^{\beta}_{\alpha}) = \phi^{\alpha}$,
2. $\bar{A}(\Psi^{\beta}_{\alpha}) = \phi^{\alpha}$.

(Note that, though the denotations of Φ^{β}_{α} and Ψ^{β}_{α} are same, their *syntactic* types are different. A universal domain allows mixing of all types; it is important not to be confused by that.)

Assume $\alpha \prec \beta$. Consider the following cases.

1. $\alpha = \omega$: Let $\Phi^{\beta}_{\alpha} = (\lambda x.\Omega) : (\beta \rightarrow \omega)$ and $\Psi^{\beta}_{\alpha} = (\lambda x.\Omega) : (\omega \rightarrow \beta)$.
2. $\alpha = \alpha_1 \rightarrow \alpha_2$, $\beta = \beta_1 \rightarrow \beta_2$ and $\alpha_1 \prec \beta_1$, $\alpha_2 \prec \beta_2$: let $\Phi^{\beta}_{\alpha} = \lambda g : \beta.(\Phi^{\beta_2}_{\alpha_2} \circ g \circ \Psi^{\beta_1}_{\alpha_1})$ and $\Psi^{\beta}_{\alpha} = \lambda f : \alpha.(\Psi^{\beta_2}_{\alpha_2} \circ f \circ \Phi^{\beta_2}_{\alpha_2})$.
3. $\alpha_1 \approx \alpha_2$, $\beta_1 \approx \beta_2$, and $\alpha_1 \prec \beta_1$: Then $\alpha_1 \rightarrow \beta_1 \approx \alpha_2 \rightarrow \beta_2$ and $\beta_1 \rightarrow \alpha_1 \approx \beta_2 \rightarrow \alpha_2$, hence we can let, $\Phi^{\beta_2}_{\alpha_2} = \Phi^{\beta_1}_{\alpha_1}$ and $\Psi^{\beta_2}_{\alpha_2} = \Psi^{\beta_1}_{\alpha_1}$.
4. transitive, reflexive closure: easy.

That $(\Psi^{\beta}_{\alpha}, \Phi^{\beta}_{\alpha})$ is indeed an injection-projection pair with the above mentioned property, is left as an easy exercise.

For any $t : \beta$, if $\alpha \prec \beta$, define its syntactic α-projection $\lfloor t \rfloor_{\alpha} \in \bar{\mathcal{L}}^{\alpha}$ as:

$$\lfloor t \rfloor_{\alpha} = (\Phi^{\beta}_{\alpha} t).$$

Then $\bar{A}(\lfloor t \rfloor_{\alpha}) = \phi^{\alpha}(\bar{A}(t))$. Also, if $\alpha \approx \beta$ then $\lfloor t \rfloor_{\alpha} \overset{\mathcal{L}}{\simeq} t$.

Similarly for any $s : \alpha$, if $\alpha \prec \beta$, define its syntactic β-injection $\lceil s \rceil^{\beta} \in \bar{\mathcal{L}}^{\beta}$ as:

$$\lceil s \rceil^{\beta} = (\Psi^{\beta}_{\alpha} s).$$

Then $\bar{A}(\lceil t\rceil^{\beta}) = \bar{A}(t)$. If $\alpha \approx \beta$ then $\lceil s\rceil^{\beta} \overset{\bar{\mathcal{L}}}{\simeq} s$.

For any $\tau \in \bar{\Gamma}$, define:

$$\tau\downarrow = \{\alpha \mid \alpha \in \Gamma \text{ and } \alpha \prec \tau\}.$$

Lemma 9.1: For any $\tau \in \bar{\Gamma}$, $\tau\downarrow$ is directed.

Proof: Suppose this is not the case. Let $\alpha \in \Gamma$ be a nonreflexive type of minimal length for which there exist $\tau \in \bar{\Gamma}$ and $\beta \in \Gamma$ such that:

1. $\alpha, \beta \in \tau\downarrow$,
2. $\alpha \sharp \beta$, i.e. α and β do not have an upper bound in $\tau\downarrow$.

We derive a contradiction. Consider three cases.

1. $\alpha = \omega$: This is not possible, as in that case $\alpha, \beta \prec \beta \in \tau\downarrow$.
2. $\alpha = \kappa$, for some ground type κ. But then as $\kappa = \alpha \in \tau\downarrow$, this means $\tau \approx \kappa$, and hence $\beta = \kappa$. Again $\alpha, \beta \prec \kappa \in \tau\downarrow$, which is a contradiction.
3. $\alpha = \alpha_1 \rightarrow \alpha_2$: Then β is also of the form $\beta_1 \rightarrow \beta_2$ for some β_1 and β_2, because otherwise the minimality of α is contradicted. Also, since $\alpha_1 \rightarrow \alpha_2 = \alpha \prec \tau$, $\tau \approx \tau_1 \rightarrow \tau_2$, for some $\tau_1, \tau_2 \in \bar{\Gamma}$. As $\alpha, \beta \prec \tau$, we conclude: $\alpha_1, \beta_1 \in \tau_1\downarrow$ and $\alpha_2, \beta_2 \in \tau_2$. If both the pairs (α_1, β_1) and (α_2, β_2) are consistent (i.e. have an upper bound) then so is (α, β), hence, without loss of generality, assume that $\alpha_1 \sharp \beta_1$ in $\tau_1\downarrow$. But, as α_1 is of smaller length than α, we have a contradiction to the minimality of α.

$\diamond$

As $\alpha \prec \beta$ implies $\phi^{\alpha} \sqsubseteq \phi^{\beta}$, by the above lemma $\{\phi^{\alpha} \mid \alpha \in \tau\downarrow\}$ is directed for every $\tau \in \bar{\Gamma}$, moreover it is easy to show that:

$$\phi^{\tau} = \{\phi^{\alpha} \mid \alpha \in \tau\downarrow\}.$$

For every $t \in \bar{\mathcal{L}}$ we inductively define $t\downarrow \subseteq \mathcal{L}$, the set of nonreflexive approximations to t:

1. $b\downarrow = \{b\}$, if b is a basic ground constant,
2. $f\downarrow = \{f\}$, if f is a first order function constant,
3. $K\downarrow = \{K : \alpha \mid \alpha \in \tau\downarrow\}$, if $K \in \bar{\mathcal{L}}^{\tau}$,
4. $S\downarrow = \{S : \alpha \mid \alpha \in \tau\downarrow\}$, if $S \in \bar{\mathcal{L}}^{\tau}$,
5. $Y\downarrow = \{Y : \alpha \mid \alpha \in \tau\downarrow\}$, if $Y \in \bar{\mathcal{L}}^{\tau}$,
6. $(rs)\downarrow = \{uv \mid u : \alpha' \rightarrow \beta', v : \alpha', u \in r\downarrow, v \in s\downarrow\}$.

We have the following approximation continuity result:

Lemma 9.2: For every $t \in \bar{\mathcal{L}}$, $\bar{A}(t) = \bigsqcup_{w \in t\downarrow}^{\infty} A(w)$.

Proof: Let $t \in \bar{\mathcal{L}}$. It is obvious that $\bar{A}(t) \sqsupseteq \bigsqcup_{w \in t\downarrow}^{\infty} A(w)$. Hence, we need to prove only the other inequality.

We consider only the most difficult case: $t = rs$, where $r \in \bar{\mathcal{L}}^{\alpha \rightarrow \beta}$ and $s \in \bar{\mathcal{L}}^{\alpha}$.

Suppose $(u : \sigma \to \tau) \in r\downarrow$, where $\sigma \in \alpha\downarrow$, $\tau \in \beta\downarrow$, and $(v : \sigma') \in s\downarrow$. Then, as $\alpha\downarrow$ is directed, there exists $\sigma'' \in \alpha\downarrow$, such that $\sigma, \sigma' \prec \sigma''$. Define $\langle uv\rangle \in (rs)\downarrow$ as $\langle uv\rangle = \lceil u\rceil^{\sigma''}{}^{,\tau}\lceil v\rceil^{\sigma''}$. Since, by the induction hypothesis, $\bar{A}(r) = \bigsqcup_{u \in r\downarrow}^{\infty} A(u)$ and $\bar{A}(s) = \bigsqcup_{v \in s\downarrow}^{\infty} A(s)$, we conclude:

$$\bar{A}(rs) = (\bar{A}r)(\bar{A}s) = (\bigsqcup_{u \in r\downarrow}^{\infty} Au)(\bigsqcup_{v \in s\downarrow}^{\infty} Av) = \bigsqcup_{u \in r\downarrow, v \in s\downarrow}^{\infty} A\langle uv\rangle \sqsubseteq \bigsqcup_{w \in t\downarrow}^{\infty} A(w).$$

◇

Call $t \in \bar{\mathcal{L}}$ finite if $\bar{A}(t)$ is finite.

Lemma 9.3: Let $\alpha \in \Gamma$. Then, for every finite $t \in \bar{\mathcal{L}}^{\alpha}$, there exists an $s \in \mathcal{L}^{\alpha}$ such that $\bar{A}(t) = A(s)$.

Proof: By approximation continuity, $\bar{A}(t) = \bigsqcup_{w \in t\downarrow}^{\infty} A(w)$. As t is finite, there exists some $(w : \beta) \in t\downarrow$, where $\beta \prec \alpha$, such that $\bar{A}(t) = A(w)$. Let $s = \lceil w\rceil^{\alpha}$.

◇

Open question: Does there exist such an s even when t is not finite? Probably not.

Let $t \in \bar{\mathcal{L}}^{\alpha}$, where $\alpha \in \Gamma$. Then, for every i, $\lfloor t\rfloor_i$ is finite. Hence by the above lemma there exists some $s \in \mathcal{L}^{\alpha}$ such that $\bar{A}(\lfloor t\rfloor_i) = A(s)$. Define $\langle t\rangle_i$ as:

$$\langle t\rangle_i = s.$$

(Which s one chooses is immatterial. Just choose one arbitarily using Axiom of Choice.)

Lemma 9.4: If $d \in D_i^{\tau}$, where $\tau \in \Gamma$, then

1. $(d, t) \in \bar{\Theta}^{\tau}$ implies $(d, \langle t\rangle_i) \in \Theta^{\tau}$,
2. $(d, s) \in \Theta^{\tau}$ implies $(d, \lfloor s\rfloor_i) \in \bar{\Theta}^{\tau}$ and, hence, $(d, s) \in \bar{\Theta}^{\tau}$.

Proof: If $\tau = \omega$ then the lemma is obvious. Otherwise let $\tau = (\tau_1, \ldots, \tau_n, \kappa)$. By induction hypothesis, assume that the lemma holds for all τ_j.

(1) Assume $(d, t) \in \bar{\Theta}^{\tau}$. Then for all $(c_j, s_j) \in \Theta^{\tau_j}$, by induction hypothesis, $(c_j, \lfloor s_j\rfloor_i) \in \bar{\Theta}^{\tau_j}$ and hence:

$$\begin{array}{rcll} dc_1 \ldots c_n & \sqsubseteq & \bar{O}(t\lfloor s_1\rfloor_i \ldots \lfloor s_n\rfloor_i) & \\ & = & (\bar{A}t)(\bar{A}\lfloor s_1\rfloor_i) \ldots (\bar{A}\lfloor s_n\rfloor_i) & \text{by the semantic equivalence between } \bar{A} \text{ and } \bar{O}. \end{array}$$

This means:

$$\begin{array}{rcll} dc_1 \ldots c_n & = & \phi_i^{\kappa}(dc_1 \ldots c_n) & \text{as } d \in D_i^{\tau} \\ & \sqsubseteq & \phi_i^{\kappa}((\bar{A}t)(\bar{A}\lfloor s_1\rfloor_i) \ldots (\bar{A}\lfloor s_n\rfloor_i)) & \\ & = & (\phi_i^{\tau}(\bar{A}t))(\bar{A}\lfloor s_1\rfloor_i) \ldots (\bar{A}\lfloor s_n\rfloor_i) & \text{as } \bar{A}(\lfloor s_j\rfloor_i) \in D_i^{\tau_j} \text{ for all } j \\ & = & (A\langle t\rangle_i)(A\lfloor s_1\rfloor_i) \ldots (A\lfloor s_n\rfloor_i) & \text{as } \lfloor s_1\rfloor_i, \ldots, \lfloor s_n\rfloor_i \in \mathcal{L} \\ & \sqsubseteq & (A\langle t\rangle_i)(As_1) \ldots (As_n) & \\ & = & O(\langle t\rangle_i s_1 \ldots s_n) & \text{by semantic equivalence.} \end{array}$$

Hence $(d, \langle t\rangle_i) \in \Theta^{\tau}$.
(2) Similar.

◇

Corollary 9.5: If $\tau \in \Gamma$, and $d_1, d_2 \in D^{\tau}_i$ then

$$d_1 \underset{\sim}{\bar{\sqsubseteq}} d_2 \text{ iff } d_1 \underset{\sim}{\sqsubseteq} d_2.$$

Proof: We only prove that $d_1 \underset{\sim}{\sqsubseteq} d_2$ implies $d_1 \underset{\sim}{\bar{\sqsubseteq}} d_2$, the proof in the other direction being similar.

Suppose $d_1 \underset{\sim}{\sqsubseteq} d_2$. Then

$$\begin{array}{lll} & (d_2, t) \in \bar{\Theta} & \\ \Rightarrow & (d_2, \langle t\rangle_i) \in \Theta & \text{by the above lemma} \\ \Rightarrow & (d_1, \langle t\rangle_i) \in \Theta & \text{as } d_1 \underset{\sim}{\sqsubseteq} d_2 \\ \Rightarrow & (d_1, \langle t\rangle_i) \in \bar{\Theta} & \text{by the above lemma} \\ \Rightarrow & (d_1, t) \in \bar{\Theta} & \text{as } \langle t\rangle_i \underset{\sim}{\overset{\mathcal{L}}{\sqsubseteq}} t. \end{array}$$

Hence $d_1 \underset{\sim}{\bar{\sqsubseteq}} d_2$.

◇

If, $\tau \in \Gamma$ and $d \in D^{\tau}_i$, then by the above corollary,

$$[d]^{\tau}_i = [\bar{d}]^{\tau}_i. \qquad (8)$$

Hence, $\bar{Q}^{\tau}_i(d) = max[\bar{d}]^{\tau}_i = max[d]^{\tau}_i = Q^{\tau}_i(d)$. This means:

$$\bar{Q}^{\tau}_i = Q^{\tau}_i, \text{ for every } \tau \in \Gamma.$$

By continuity,

$$\bar{Q}^{\tau} = Q^{\tau}, \text{ for every } \tau \in \Gamma.$$

Note that if $\tau \in \Gamma$ and d is finite then $d \in D^{\tau}_j$, for some j, and hence by (8):

$$[d]^{\tau}_i = [\bar{d}]^{\tau}_i, \text{ for all } i \geq j.$$

But this does *not* necessarily mean that $[d]^{\tau} = [\bar{d}]^{\tau}$, as $[d]^{\tau}$ or $[\bar{d}]^{\tau}$ might contain some element c none of whose finite approximations belong to them. Hence,

Open Question: Is $[d]^{\tau} = [\bar{d}]^{\tau}$ for every *finite* $d \in D^{\tau}$?

Better still:

Is $[d]^{\tau} = [\bar{d}]^{\tau}$ for *every* $d \in D^{\tau}$?

If the answer to the second question is negative, we have an interesting situation where the quotient spaces $D^{\tau}_{\sim}, D^{\tau}_{\bar{\sim}}$ might not be equal but their closures $\bar{Q}^{\tau}, Q^{\tau}$ are!

10. Relationship Between Reflexive Quotient Closures

In this section we investigate the relationship between various reflexive quotient closures $\bar{Q}_i^\tau$. We show that $\bar{Q}_i^\alpha \prec \bar{Q}_i^\beta$, if $\alpha \prec \beta$.

If $\alpha \prec \beta$ let us define

$$\lceil \bar{\Theta}^\alpha \rceil^\beta = \{(d, \lceil t \rceil^\beta) \mid (d,t) \in \bar{\Theta}^\alpha\},$$

and

$$\lfloor \bar{\Theta}^\beta \rfloor_\alpha = \{(t, \lfloor t \rfloor_\alpha) \mid (d,t) \in \bar{\Theta}^\beta\}.$$

Lemma 10.1: If $\alpha \prec \beta$ then

1. $\lceil \bar{\Theta}^\alpha \rceil^\beta \subseteq \bar{\Theta}^\beta$,
2. $\lfloor \bar{\Theta}^\beta \rfloor_\alpha \subseteq \bar{\Theta}^\alpha$.

Proof: We only show 1. The proof of 2. is analogous.

Consider three cases.

(1) $\alpha = \omega$: Obvious.

(2) $\alpha = \alpha_1 \to \alpha_2$, $\beta = \beta_1 \to \beta_2$, and $\alpha_1 \prec \beta_1$, $\alpha_2 \prec \beta_2$: Assume, by induction hypothesis, that the lemma holds for α_j and β_j, $j = 1, 2$. Let $(d,t) \in \bar{\Theta}^\alpha$. We have to show that $(d, \lceil t \rceil^\beta) \in \bar{\Theta}^\beta = \bar{\Theta}^{\beta_1 \to \beta_2}$.

Consider an arbitrary $(c,s) \in \bar{\Theta}^{\beta_1}$. Then, remembering that $d \in D^{\alpha_1 \to \alpha_2}$,

$$(dc, \lceil t \rceil^\beta s) = (dc, \lceil t \rceil^{\beta_1 \to \beta_2}) = (d(\phi^{\alpha_1} c), \Psi^{\beta_2}(t(\Phi^{\alpha_1} s))). \tag{9}$$

Now:

$$\begin{array}{lll}
 & (c,s) \in \bar{\Theta}^{\beta_1} & \\
\Rightarrow & (\phi^{\alpha_1} c, \Phi^{\alpha_1} s) \in \bar{\Theta}^{\alpha_1} & \text{by induction hypothesis,} \\
\Rightarrow & (d(\phi^{\alpha_1} c), t(\Phi^{\alpha_1} s)) \in \bar{\Theta}^{\alpha_2} & \text{as } (d,t) \in \bar{\Theta}^\alpha = \bar{\Theta}^{\alpha_1 \to \alpha_2}, \\
\Rightarrow & (d(\phi^{\alpha_1} c), \Psi^{\beta_2}(t(\Phi^{\alpha_1} c))) \in \bar{\Theta}^{\beta_2} & \text{by induction hypothesis,} \\
\Rightarrow & (dc, \lceil t \rceil^\beta s) \in \bar{\Theta}^{\beta_2} & \text{by (9).}
\end{array}$$

Hence, for all $(c,s) \in \bar{\Theta}^{\beta_1}$, $(dc, \lceil t \rceil^\beta s) \in \bar{\Theta}^{\beta_2}$. This means $(d, \lceil t \rceil^\beta) \in \bar{\Theta}^\beta$.

(3) $\alpha \approx \alpha'$, $\beta \approx \beta'$, and $\alpha' \prec \beta'$: But then, $\bar{\Theta}^\alpha = \bar{\Theta}^{\alpha'}$ and $\bar{\Theta}^\beta = \bar{\Theta}^{\beta'}$, hence the result follows.

◇

Lemma 10.2: If $\alpha \prec \beta$ and $d_1, d_2 \in D^\alpha$, then

$$d_1 \underset{\sim_\alpha}{\bar{\sqsubset}} d_2 \text{ iff } d_1 \underset{\sim_\beta}{\bar{\sqsubset}} d_2.$$

Proof: Remember that $d_1, d_2 \in D^\alpha$ implies $d_1, d_2 \in D^\beta$. We shall only prove that $d_1 \underset{\sim_\alpha}{\bar{\sqsubset}} d_2$ implies $d_1 \underset{\sim_\beta}{\bar{\sqsubset}} d_2$; the proof in the other direction is similar.

If $d_1 \underset{\sim}{\bar{\sqsubseteq}}_{\alpha} d_2$ then:

$$\begin{array}{lll}
 & (d_2, t) \in \bar{\Theta}^{\beta} & \\
\Rightarrow & (d_2, \lfloor t \rfloor_{\alpha}) \in \bar{\Theta}^{\alpha} & \text{by the above lemma} \\
\Rightarrow & (d_1, \lfloor t \rfloor_{\alpha}) \in \bar{\Theta}^{\alpha} & \text{as } d_1 \underset{\sim}{\bar{\sqsubseteq}}_{\alpha} d_2 \\
\Rightarrow & (d_1, \lceil \lfloor t \rfloor_{\alpha} \rceil^{\beta}) \in \bar{\Theta}^{\beta} & \text{by the above lemma} \\
\Rightarrow & (d_1, t) \in \bar{\Theta}^{\beta} & \text{as } \lceil \lfloor t \rfloor_{\alpha} \rceil^{\beta} \overset{\mathcal{L}}{\underset{\sim}{\sqsubseteq}} t.
\end{array}$$

Hence $d_1 \underset{\sim}{\bar{\sqsubseteq}}_{\beta} d_2$.

◇

Corollary 10.3: If $\alpha \prec \beta$,

1. $d \in D^{\alpha}$ implies $[\bar{d}]^{\alpha} \subseteq [\bar{d}]^{\beta}$,
2. $d \in D_i^{\alpha}$ implies $[\bar{d}]_i^{\alpha} \subseteq [\bar{d}]_i^{\beta}$.

◇

Now if $\alpha \prec \beta$ and $d \in D_i^{\alpha}$ then from the above corollary it easily follows that:

$$max[\bar{d}]_i^{\alpha} = \phi_i^{\alpha}(max[\bar{d}]_i^{\beta}).$$

From which, one easily obtains:

$$\bar{F}_i^{\alpha} \prec \bar{F}_i^{\beta}.$$

(With the obvious abuse of notation, as strictly speaking, neither $\bar{F}_i^{\alpha}$ nor $\bar{F}_i^{\beta}$ is a retract.) By continuity,

$$\bar{Q}_i^{\alpha} \prec \bar{Q}_i^{\beta}.$$

Applying continuity once more,

$$\bar{Q}^{\alpha} \prec \bar{Q}^{\beta}.$$

11. Continuity Argument

In this section we shall show, using continuity argument, that $\bar{M}^Q$ is an extensional, fully abstract, algebraic model of $\bar{\mathcal{L}}$.

Before that let us ask ourselves two questions:

1. What is the precise relationship between $\bar{Q}_i^\tau$ and $\bar{Q}_j^\tau$, if $i \leq j$?
2. What is the precise relationship between $\bar{Q}_i^\alpha$ and $\bar{Q}_i^\beta$, if $\alpha \prec \beta$?

If we assume that i increases in the horizontal direction and types increase in the vertcal direction, the first question enquires about the relationship in the horizontal direction and the second question enquires about the relationship in the vertical direction. As it will turn out, both the relationships are exactly similar, which is very pleasing.

The answer to the first question is easy. We know now that $\bar{Q}_i^\alpha = Q_i^\alpha$, if $\alpha \in \Gamma$. Hence from what we have already proved for the nonreflexive case, it follows that the family $\{\bar{Q}_i^\alpha \mid \alpha \in \Gamma\} = \{Q_i^\alpha \mid \alpha \in \Gamma\}$ is closed under application and

$$\{\bar{Q}_i^\alpha \mid \alpha \in \Gamma\} \lhd \{\bar{Q}_j^\alpha \mid \alpha \in \Gamma\}, \text{ if } i \leq j.$$

Note that this implies not only that $\bar{Q}_i^\alpha \prec \bar{Q}_j^\alpha$, but also that the application remains invariant under this embedding. It is easy to prove that for every $\tau \in \bar{\Gamma}$:

$$\bar{Q}_i^\tau = \bigsqcup_{\alpha \in \tau\downarrow}^{\infty} \bar{Q}_i^\alpha = \bigsqcup_{\alpha \in \tau\downarrow}^{\infty} Q_i^\alpha.$$

Hence, using the continuity argument as in Section 5, it follows that $\{\bar{Q}_i^\tau \mid \tau \in \bar{\Gamma}\}$ is closed under application and

$$\{\bar{Q}_i^\tau \mid \tau \in \bar{\Gamma}\} \lhd \{\bar{Q}_j^\tau \mid \tau \in \bar{\Gamma}\}, \text{ if } i \leq j.$$

We have already partly answered the second question: we showed in the previous section that $\bar{Q}_i^\alpha \prec \bar{Q}_i^\beta$, if $\alpha \prec \beta$. But we still have to show that this vertical embedding behaves nicely with respect to the application.

Lemma 11.1: Let $\tau_1 = \alpha_1 \to \beta_1$, $\tau_2 = \alpha_2 \to \beta_2$, where $\alpha_1 \prec \beta_1$ and $\alpha_2 \prec \beta_2$. For a fixed i, let $d \in \bar{Q}_i^{\tau_2}$ and $c \in \bar{Q}_i^{\alpha_1}$. Then

$$\bar{Q}_i^{\beta_1}(d(\bar{Q}_i^{\alpha_2}c)) = (\bar{Q}_i^{\tau_1}d)c.$$

Proof:It suffices to prove the lemma when $\alpha_j, \beta_j \in \Gamma$; in the general case the lemma is proved by continuity argument. But then $\bar{Q}_i^{\alpha_1}, \bar{Q}_i^{\beta_1} \ldots$ are all finite, hence the proof similar to the one of Lemma 5.1 works. (Finiteness of $\bar{Q}_i^{\alpha_1}, \ldots$ is required in the definition of the step function.)

◇

It can be easily shown from the above lemma that the following diagram commutes:

$$\begin{tikzcd}
Q_i^{\alpha_2 \to \beta_2} \times Q_i^{\alpha_2} \arrow[r, "\cdot"] & Q_i^{\beta_2} \arrow[d, "Q_i^{\beta_1}"] \\
Q_i^{\alpha_1 \to \beta_1} \times Q_i^{\alpha_1} \arrow[u, "{(Q_i^{\alpha_2 \to \beta_2}, Q_i^{\alpha_2})}"] \arrow[r, "\cdot"] & Q_i^{\beta_1}
\end{tikzcd}$$

This says that the application remains invariant under vertical embedding too.

Define $(\alpha, i) \leq (\beta, j)$, if $\alpha \prec \beta$ and $i \leq j$. Then combining the horizontal and vertical embeddings we get a diagonal embedding:

$$\bar{Q}^\alpha_i \prec \bar{Q}^\beta_j, \text{ if } (\alpha, i) \leq (\beta, j),$$

and the application remains invariant under the diagonal embedding too. We now have a nice commutative diagram:

$$\begin{tikzcd}
\bar{Q}^\beta_i \arrow[r] & \bar{Q}^\beta_j \\
\bar{Q}^\alpha_i \arrow[r] \arrow[u] \arrow[ur] & \bar{Q}^\alpha_j \arrow[u]
\end{tikzcd}$$

where $(\alpha, i) \leq (\beta, j)$ and the arrows indicate embeddings.

As the embeddings behave nicely in all directions, the reader can easily prove, by the continuity argument, that $\{\bar{Q}^\tau \mid \tau \in \bar{\Gamma}\}$ is closed under application and, moreover, it is extensional and algebraic. In fact $\{\bar{Q}^\tau \mid \tau \in \bar{\Gamma}\}$ is the limit of the horozontal embedding sequence $\{\bar{Q}^\tau_1 \mid \tau \in \bar{\Gamma}\} \lhd \{\bar{Q}^\tau_2 \mid \tau \in \bar{\Gamma}\} \lhd \ldots$ It can also be regarded as the 'vertical closure' of the family $\{\bar{Q}^\alpha \mid \alpha \in \Gamma\} = \{Q^\alpha \mid \alpha \in \Gamma\}$. Thus there is more than one way of proving the above assertion!

For every $t \in \bar{\mathcal{L}}$,

$$\begin{aligned}
\bar{A}^Q(t) &= \bar{Q}(\bar{A}t) \\
&= \Big(\bigsqcup^{\infty}_{\alpha \in \tau\downarrow} Q^\alpha\Big)\Big(\bigsqcup^{\infty}_{w \in t\downarrow} A(w)\Big) \\
&= \bigsqcup^{\infty}_{w \in t\downarrow} Q \circ A(w) \\
&= \bigsqcup^{\infty}_{w \in t\downarrow} A^Q(w).
\end{aligned}$$

Hence, as A^Q is a homomorphism, so is $\bar{A}^Q$.

We have proved that

$\bar{M}^Q = (\bar{Q}^\tau, \cdot, \bar{A}^Q)$ is an algebraic, extensional model of $\bar{\mathcal{L}}$.

We turn to full abstractness of $\bar{M}^Q$. First let us prove some lemmas.

Lemma 11.2:

1. $t \overset{\bar{\mathcal{L}}}{\underset{\sim}{\sqsubseteq}} s$ iff $(\bar{A}t, s) \in \bar{\Theta}$,
2. $(d, t) \in \bar{\Theta}$ iff $d \underset{\sim}{\sqsubseteq} \bar{A}(t)$,
3. $t \overset{\bar{\mathcal{L}}}{\underset{\sim}{\sqsubseteq}} s$ iff $\bar{A}(t) \underset{\sim}{\sqsubseteq} \bar{A}(s)$.

Proof: As the proof is very similar to the one in the nonrelexive case, we shall only prove that $(\bar{A}t, s) \in \bar{\Theta}$ implies $t \overset{\bar{\mathcal{L}}}{\underset{\sim}{\sqsubseteq}} s$.

Suppose $(\bar{A}t, s) \in \bar{\Theta}^{\tau}$. If $\tau \approx (\tau_1, \ldots, \tau_n.\kappa)$, then by Corollary 7.2, $\bar{\Theta}^{\tau} = \bar{\Theta}^{(\tau_1,\ldots,\tau_n.\kappa)}$ and, hence, $(\bar{A}t, s) \in \bar{\Theta}^{(\tau_1,\ldots,\tau_n.\kappa)}$. This implies that for all $(t_1 : \tau_1), \ldots, (t_n : \tau_n)$:

$$\begin{array}{rcll} \bar{O}(tt_1 \ldots t_n) & = & \bar{A}(tt_1 \ldots t_n) & \\ & = & (\bar{A}t)(\bar{A}t_1) \ldots (\bar{A}t_n) & \\ & \sqsubseteq & \bar{O}(st_1 \ldots t_n) & \quad \text{as } (\bar{A}t_j, t_j) \in \bar{\Theta}^{\tau_j}. \end{array}$$

Hence $t \overset{\mathcal{L}}{\underset{\sim}{\sqsubseteq}} s$.

◇

If $\alpha \in \tau\downarrow$ then, for every $t : \tau$, define $\langle t \rangle_{(\alpha,i)} = \langle \lfloor t \rfloor_{\alpha} \rangle_i$.

Lemma 11.3: If $t, s : \tau$ then $t \overset{\mathcal{L}}{\underset{\sim}{\sqsubseteq}} s$ implies $\langle t \rangle_{(\alpha,i)} \overset{\mathcal{L}_i}{\underset{\sim}{\sqsubseteq}} \langle s \rangle_{(\alpha,i)}$ for every $\alpha \in \tau\downarrow$ and i.

Proof: similar to the one of Lemma 3.1.

◇

Now if $\bar{A}^Q(t) \sqsubseteq \bar{A}^Q(s)$, then:

$$\bar{A}(t) \sqsubseteq \bar{A}^Q(t) \sqsubseteq \bar{A}^Q(s) = \bar{Q}(\bar{A}s) \mathrel{\bar{\simeq}} \bar{A}(s).$$

Hence $\bar{A}(t) \mathrel{\bar{\simeq}} \bar{A}(s)$, which, by Lemma 11.2, implies $t \overset{\mathcal{L}}{\underset{\sim}{\sqsubseteq}} s$.

On the other hand, if $(t : \tau) \overset{\mathcal{L}}{\underset{\sim}{\sqsubseteq}} (s : \tau)$ then, by Lemma 11.3, for all $\alpha \in \tau\downarrow$ and i, $\langle t \rangle_{(\alpha,i)} \overset{\mathcal{L}_i}{\underset{\sim}{\sqsubseteq}} \langle s \rangle_{(\alpha,i)}$, which, as every M_i^Q is fully abstract, implies $A^Q(\langle t \rangle_{(\alpha,i)}) \sqsubseteq A^Q(\langle s \rangle_{(\alpha,i)})$. Hence:

$$\begin{array}{rcl} \bar{A}^Q(t) & = & \displaystyle\bigsqcup_{\alpha \in \tau\downarrow, i}^{\infty} A^Q \langle t \rangle_{(\alpha,i)} \\ & \sqsubseteq & \displaystyle\bigsqcup_{\alpha \in \tau\downarrow, i}^{\infty} A^Q \langle s \rangle_{(\alpha,i)} \\ & = & \bar{A}^Q(s). \end{array}$$

Thus indeed,

$\bar{M}^Q$ is fully abstract.

One proves, just as in the nonreflexive case, that Y has the standard denotation in $\bar{M}^Q$ and that $\bar{M}^Q$ is a β-model.

As $\bar{A}^Q$ is shown to be a homomorphism, its equivalent denotational definition can now be given:

1. $\bar{A}^Q(b) = \bar{Q}(b)$ where b is a basic ground type constant.
2. $\bar{A}^Q(f) = \bar{Q}(f)$ where f is a basic first order function constant.
3. $\bar{A}^Q(S) = \bar{Q}(\lambda x, y, z.(xz)(yz))$
4. $\bar{A}^Q(K) = \bar{Q}(\lambda x, y.x)$
5. $\bar{A}^Q(Y) = \bar{Q}(\bigsqcup_{n=0}^{\infty} \lambda f.f^n(\bot))$

6. $\bar{A}^Q(rs) = (\bar{A}^Q r)(\bar{A}^Q s)$ where r and s are of the appropriate types.

We have proved the second main result of this paper:

Theorem 11.4: $\bar{M}^Q = (\bar{Q}^\tau, \cdot, \bar{A}^Q)$ is a fully abstract, extensional, algebraic, β-model of $\bar{\mathcal{L}}$. Moreover Y has the standard interpretation in $\bar{M}^Q$ and the following diagram commutes.

$$\begin{tikzcd}
\bar{\mathcal{L}} \arrow[r, "\bar{A}"] \arrow[dr, "\bar{A}^Q"'] & \bar{M} \arrow[d, "Q"] \\
& \bar{M}^Q
\end{tikzcd}$$

Further $\bar{M}^Q$ is just an extension of M^Q, i.e.

1. $\bar{Q}^\alpha = Q^\alpha$, if $\alpha \in \Gamma$,
2. $\bar{A}^Q(t) = \bar{A}^Q(t)$, if $t \in \mathcal{L}$.

◇

Finally one can prove from the *existence* of $\bar{M}^Q$:

$$t \overset{\mathcal{L}}{\underset{\sim}{\sqsubseteq}} s \text{ iff } \bar{O}(C[t]) \sqsubseteq \bar{O}(C[s]) \text{ for all ground contexts } C[\ ].$$

12. Diagonalization

Now we must face the question: why is it that we gave preferene to $\sqsubseteq$ over $\sqsupseteq$ or $=$ in the definition of Θ or $\bar{\Theta}$ predicates?

That we could not use $=$ is obvious, because we want the property:

$$d_1 \sqsubseteq d_2 \text{ implies } d_1 \underset{\sim}{\sqsubseteq} d_2.$$

But what about $\sqsupseteq$? (Then we shall have to replace *max* with *min* everywhere in the treatment.) It can be shown that Θ and $\bar{\Theta}$ predicates exist for *any* operational semantics O or $\bar{O}$. (This does *not* mean that our theory goes through for any operational semantics, because we used the equivalence between the denotational and operational semantics on innumerable occasions.) We shall show, using diagonalization, that this is not the case when $\sqsubseteq$ is replaced by $\sqsupseteq$. This provides, at least, a partial justification for our preference.

Suppose κ is a nontrivial ground type, i.e. D^κ has atleast two elements. Choose an $\bar{O}$ such that, for every $t : \kappa$,

$$\bar{O}(t) = b \neq \bot,$$

for some $b \in D^\kappa$. Let $\tau = !\alpha.(\alpha \rightarrow \kappa)$. Now if $\sqsubseteq$ were replaced by $\sqsupseteq$ in the definitions of the $\bar{\Theta}$ predicates, then the 'new' $\bar{\Theta}^\tau$ should satisfy the recursive equation:

$$\bar{\Theta}^\tau = \{(d,t) \mid \text{ for all}(c,s) \in \bar{\Theta}^\tau.dc \sqsupseteq \bar{O}(ts)\}.$$

We show that the above equation has no solution! As $\bar{O}(r) = b$, for every $r : \kappa$, this reduces to showing that the equation

$$\bar{\Sigma}^\tau = \{d \in D^\tau \mid \text{ for all } c \in \bar{\Sigma}^\tau.dc \sqsupseteq b\} \qquad (10)$$

has no solution.

Suppose it had a solution $\bar{\Sigma}^\tau$. Let $f{:}D^\tau = \lambda x.xx$. Consider two cases.

1. $f \in \bar{\Sigma}^\tau$: But then $ff \sqsupseteq b \neq \bot$. However, by the argument similar to the one in [Park] one can show that $ff = (\lambda x.xx)(\lambda x.xx) = \bot$. This is a contradiction.
2. $f \notin \bar{\Sigma}^\tau$: But then for all $d \in \bar{\Sigma}^\tau$:

$$\begin{aligned} fd &= dd \\ &\sqsupseteq b \quad \text{as } d \in \bar{\Sigma}^\tau. \end{aligned}$$

Hence $f \in \bar{\Sigma}^\tau$. Again we arive at a contradiction.

We conclude that (10) has no solution.

When $\bar{O}$ is the *actual* operational semantics, we leave the above existence problem open. (A similar question was raised in [Stoy]. We have answered it partially above.)

13. Conclusion

In this paper we have provided a semantic characterization of fully abstract, extensional, algebraic models of typed lambda calculi with or without reflexive types. One pleasing aspect of the theory was that it meshed so well with the proof of the equivalence between the denotational and operational semantics. One question which needs further study is: what happens when we enrich a language? How is the fully abstract model, as constructed in this paper, of the enriched language related to the fully abstract model of the original language? We have already answered this question in one instance when we showed that the fully abstract model for the language enriched with reflexive types, $\bar{M}^Q$, is just an extension of the fully abstract model of the original language, M^Q. But this question deserves further study.

The domains used for the semantics of the 'real' programming languages are reflexive. But, as our theory works in the presence of reflexive domains, it is hoped that the above theory should work for the real programming languages too. Note that in the case of reflexive domains the definitions of inclusive predicates used for collapsing are recursive, so one must ensure that the predicates do exist. General techniques were given in [Reynolds] and [Milne] to show the existence of such predicates. Unfortunately these proofs are known to be complicated, and so there has been a tendency to skip these proofs. But in Section 12 we showed, by diagonalization, that this tendency is unjustified and dangerous because there exist nontrivial examples of inclusive predicate definitions which do not have any solution! In [Mulmuley] a general theory was given which could be used to prove the existence of the inclusive predicates, when they did exist, and which had an added advantage of being mechanizable. A system was built on top of LCF, which could (almost) *automatically* prove the existence of such inclusive predicates. The implementation is quite nontrivial. A look at any existence proof in the literature will give an idea of the amount of reasoning the system has to carry out. What is important in the present context is that, because the same predicates can be used to obtain a semantic characterization of a fully abstract model, *a large chunk of a semantic characterization proof can be automated!*

Finally it remains to be seen if this technique can be extended to deal with powerdomains.

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