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Glynn Winskel
Computer Science Department
Carnegie-Mellon University
Pittsburgh, Pa.

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by
Glynn Winskel
Department of Computer Science
Carnegie-Mellon University
Pittsburgh, Pennsylvania 15213

It is easily seen that the left–closed (=downwards–closed) subsets of a partial order form a completely distributive algebraic lattice when ordered by inclusion. Here a converse is proved; any completely distributive algebraic lattice is isomorphic to such a set of left–closed subsets of a partial order. The partial order can be recovered from the lattice as the order of the lattice restricted to its complete primes.

1 Basic Definitions.

The following definitions are well–known, see *e.g.* [Grä, CL, J].

For a partial order $\mathbf{L} = (L, \sqsubseteq)$, the *covering relation* $\prec$ is defined by

$$x \prec y \Leftrightarrow x \sqsubseteq y \ \&\ x \neq y \ \&\ (\forall z.\, x \sqsubseteq z \sqsubseteq y \Rightarrow x = z \text{ or } z = y)$$

for $x, y \in L$.

Recall a *directed set* of a partial order $(L, \sqsubseteq)$ is a non–null subset $S \subseteq L$ such that $\forall s, t \in S \exists u \in S.\ s \sqsubseteq u \ \&\ t \sqsubseteq u$.

A *complete lattice* is a partial order $\mathbf{L} = (L, \sqsubseteq)$ which has joins (=suprema=least upper bounds) $\bigsqcup X$ and meets (=infima=greatest lower bounds) $\prod X$ of arbitrary subsets X of L. We write $x \sqcup y$ for $\bigsqcup\{x, y\}$, and $x \sqcap y$ for $\prod\{x, y\}$.

An *isolated (= finite=compact)* element of a complete lattice $\mathbf{L} = (L, \sqsubseteq)$ is an element $x \in L$ such that for any directed subset $S \subseteq L$ when $x \sqsubseteq \bigsqcup S$ there is $s \in S$ such that $x \sqsubseteq s$. (In a computational framework the isolated elements are that information which a computation can realise—use or produce—in finite time—see [S].)

When there are enough isolated elements to form a basis a complete lattice is said to be *algebraic* *i.e.* an *algebraic lattice* is a complete lattice $\mathbf{L} = (L, \sqsubseteq)$ for which $x = \bigsqcup\{ e \sqsubseteq x \mid e \text{ is isolated} \}$ for all $x \in L$.

Let $\mathbf{L} = (L, \sqsubseteq)$ be a complete lattice. We are interested in these distributivity laws:

$$\prod_{i \in I} \bigsqcup_{j \in J(i)} x_{i,j} = \bigsqcup_{f \in K} \prod_{i \in I} x_{i,f(i)} \tag{1}$$

where K is the set of functions $f : I \to \bigcup_{i \in I} J(i)$ such that $f(i) \in J(i)$; when **L** satisfies (1) it is said to be *completely distributive.*

$$(\bigsqcup X) \sqcap y = \bigsqcup \{ x \sqcap y \mid x \in X \} \tag{2}$$

where $X \subseteq L$ and $y \in L$; when **L** satisfies (2) it is called a *complete Heyting algebra.*

$$(\prod X) \sqcup y = \prod \{ x \sqcup y \mid x \in X \} \tag{3}$$

where $X \subseteq L$ and $y \in L$.

$$x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z) \tag{4}$$

where $x, y, z \in L$. This finite distributive law is equivalent to its dual in a complete lattice—see [Grä]. Note that simple arguments by induction show that (4) implies finite versions of (1)—in which the indexing sets are restricted to be finite—and (2) and (3)—in which the set X is restricted to be finite.

Clearly if a complete lattice is completely distributive, *i.e.* satisfies (1), then it also satisfies (2), (3) and (4).

The following definitions are perhaps less standard. Given a partial order **P**, we shall order the set of left–closed subsets of **P** by inclusion. The points of **P** can be recovered as the complete primes in this order of left–closed subsets. (Computationally, structures like **P**, and accompanying structures of left–closed subsets, can be associated with sets of events ordered by a causal dependency relation—see [NPW, W, W1, FT].)

2 Definition. Let $\mathbf{P} = (P, \leq)$ be a partial order. A subset X of P is *left–closed* iff

$$p' \leq p \in X \Rightarrow p' \in X$$

for $p, p' \in P$.

Let X be a subset of P. Define the *left–closure* of X to be

$$[X] =_{def} \{ p' \in P \mid \exists p \in X . \, p' \leq p \}.$$

By convention we write $[p]$ for $[\{ p \}] = \{ p' \in P \mid p' \leq p \}$ when $p \in P$.

3 Definition. Let $\mathbf{L} = (L, \sqsubseteq)$ be a complete lattice. A *complete prime* of **L** is an element $p \in L$ such that

$$p \sqsubseteq \bigsqcup X \Rightarrow \exists x \in X . \, p \sqsubseteq x$$

The lattice L is *prime algebraic* iff $x = \bigsqcup \{ p \sqsubseteq x \mid p \textit{ is a complete prime} \}$, for all $x \in L$.

The definition of prime algebraic was introduced in [NPW]. However, it turns out that the concept was already familiar in another guise; for complete lattices it is equivalent to algebraicity with complete distributivity. Firstly we recall a theorem from [NPW]. A prime algebraic complete lattice can always be represented, to within isomorphism, as the lattice, ordered by inclusion, of the left-closed subsets of its complete primes.

4 Theorem.

(i) Let $\mathbf{P} = (P, \leq)$ *be a partial order. Its left-closed subsets ordered by inclusion,* $(\mathcal{L}(\mathbf{P}), \subseteq)$*, form a prime algebraic complete lattice; the complete primes of* $(\mathcal{L}(\mathbf{P}), \subseteq)$ *have the form* $[p]$ *for* $p \in P$*. The partial order* $\mathbf{P}$ *is isomorphic to* $(\{\,[p] \mid p \in P\,\}, \subseteq)$*, the restriction of the ordering on left-closed subsets to the complete primes, with isomorphism given by the map* $p \mapsto [p]$*, for* $p \in P$*.*

(ii) Let $\mathbf{L} = (L, \sqsubseteq)$ *be a prime algebraic complete lattice. Let* $\mathbf{P} = (P, \leq)$ *be the partial order consisting of the complete primes of* $\mathbf{L}$ *ordered by the restriction* $\leq = \sqsubseteq \lceil P$ *of* $\sqsubseteq$ *to* P*. Then* $\theta : (\mathcal{L}(\mathbf{P}), \subseteq) \cong \mathbf{L}$ *where* $\theta(X) = \sqcup X$ *for* $X \in \mathcal{L}(\mathbf{P})$*, with inverse* ϕ *given by* $\phi(x) = \{\, p \in P \mid p \sqsubseteq x \,\}$ *for* $x \in \mathbf{L}$*.*

Proof.

(i) Let $\mathbf{P} = (P, \leq)$ be a partial order. It is easy to see that $\mathcal{L}(\mathbf{P})$ is a complete lattice in which joins are unions and meets are intersections.

Suppose x is a complete prime of $(\mathcal{L}(\mathbf{P}), \subseteq)$. Then obviously $x = \bigcup\{\,[p] \mid p \in x\,\}$ which implies $x = [p]$ for some $p \in P$. To see the converse, consider an element of the form $[p]$, for $p \in P$. If $[p] \subseteq \bigcup X$ for $X \subseteq \mathcal{L}(\mathbf{P})$ then $p \in x$ for some $x \in X$. But x is left-closed so $[p] \subseteq x$. Thus $[p]$ is a complete prime.

It is easy to see that the map $p \mapsto [p]$, for $p \in P$, is an order isomorphism between $\mathbf{P}$ and $(\{\,[p] \mid p \in P\,\}, \subseteq)$.

(ii) Let $\mathbf{L} = (L, \sqsubseteq)$ be a prime algebraic complete lattice. Let $\mathbf{P} = (P, \leq)$ be the complete primes of $\mathbf{L}$ ordered by the restriction of $\sqsubseteq$.

Obviously the maps θ and ϕ are monotonic *i.e.* order preserving. We show they are mutual inverses and so give the required isomorphism.

Firstly we show $\theta \circ \phi = 1$. Thus we require $x = \bigsqcup\{\, p \in P \mid p \sqsubseteq x\,\}$ for all $x \in L$. But this is just the condition of prime algebraicity.

Now we show $\phi \circ \theta = 1$. Let $X \in \mathcal{L}(P, \leq)$. We require $X = \phi \circ \theta(X)$ *i.e.* $X = \{\, p \in P \mid p \sqsubseteq \bigsqcup X\,\}$. Clearly $X \subseteq \{\, p \in P \mid p \sqsubseteq \bigsqcup X\,\}$. Conversely if $p \sqsubseteq \bigsqcup X$, where p is a complete prime, then certainly $p \sqsubseteq q$ for some $q \in X$. However X is left-closed so $p \in X$, showing the converse inclusion.

Thus we have established the required isomorphism. ∎

5 Corollary. *A prime algebraic complete lattice is completely distributive (and so satisfies the distributive laws (2), (3) and (4), as well as (1)).*

Proof. The distributive laws clearly hold for left–closed subsets ordered by inclusion and these represent all the prime algebraic complete lattices to within isomorphism. ∎

The next step is to show the prime algebraic complete lattices are the completely distributive algebraic lattices. A key idea is that algebraicity implies a form of discreteness; any distinct comparable pair of elements of an algebraic lattice are separated by a covering interval. The proof uses Zorn's lemma.

6 Lemma. *Let* $\mathbf{L} = (L, \sqsubseteq)$ *be an algebraic lattice. Then*

$$\forall x, y \in \mathbf{L}.\, x \sqsubseteq y \;\&\; x \neq y \Rightarrow \exists z, z' \in \mathbf{L}.\, x \sqsubseteq z \prec z' \sqsubseteq y.$$

Proof. Suppose x, y are distinct elements of $\mathbf{L}$ such that $x \sqsubseteq y$. Because $\mathbf{L}$ is algebraic there is an isolated element b such that $b \not\sqsubseteq x \;\&\; b \sqsubseteq y$. By Zorn's lemma there is a maximal chain C of elements above x and strictly below $x \sqcup b$. As b is isolated, from the construction of C we must have $x \sqsubseteq \bigsqcup C \prec x \sqcup b \sqsubseteq y$. ∎

In proving the next theorem we use such coverings to construct complete primes of a lattice. The distributive laws (2) and (3)—implied of course by (1)—make it possible to find $\sqsubseteq$- minimum coverings which correspond to complete primes. Algebraicity ensures there are enough covering intervals, and so complete primes, for the lattice to be prime algebraic.

7 Theorem. *Let* $\mathbf{L}$ *be a complete lattice. Then* $\mathbf{L}$ *is prime algebraic iff it is algebraic and satisfies the distributive laws (2) and (3).*

Proof.

"*only if*":

Let $\mathbf{L}$ be a prime algebraic complete lattice. Let $\mathbf{P}$ be the ordering of $\mathbf{L}$ restricted to its complete primes. By the previous theorem we know $\mathbf{L} \cong \mathcal{L}(\mathbf{P})$ so it is sufficient to prove properties for $\mathcal{L}(\mathbf{P})$. We have already seen the distributivity laws follow from the corresponding laws for sets.

The isolated elements $\mathcal{L}(\mathbf{P})$ are easily shown to be precisely the left–closures of finite subsets of P. Suppose $x \in \mathcal{L}(\mathbf{P})$ is isolated. Obviously $x = \bigcup\{\, [X] \mid X \subseteq^{\text{fin}} x \,\}$. But the set $\{\, [X] \mid X \subseteq^{\text{fin}} x \,\}$ is clearly directed so, because x is isolated, $x = [X]$ for some finite set $X \subseteq P$. Conversely, it is clear that an element of the form $[X]$, for a finite $X \subseteq P$, is necessarily isolated; if $[X] \subseteq \bigcup S$ for a directed subset S of $\mathcal{L}(\mathbf{P})$ then X, and so $[X]$, is included in the union of a finite subset of S, and so in an element of S. Clearly now every element of $\mathcal{L}(\mathbf{P})$ is the least upper bound of the isolated elements below it, making $\mathcal{L}(\mathbf{P})$ algebraic.

Thus $\mathbf{L}$ is an algebraic lattice satisfying the distributive laws (1), (2), (3) and (4).

"if":

Let $\mathbf{L} = (L, \sqsubseteq)$ be an algebraic lattice satisfying the distributive laws (2) and (3).

Let $x \prec x'$ in $\mathbf{L}$. Define $pr[x, x'] = \prod\{ y \in L \mid x' \leq x \sqcup y \}$. We show $p = \mathrm{pr}[x, x']$ is a complete prime of $\mathbf{L}$. Note first that $x \sqcup p = \prod\{ x \sqcup y \mid x' \sqsubseteq x \sqcup y \} = x'$ by distributive law (3). Now suppose $p \sqsubseteq \bigsqcup Z$ for some $Z \subseteq L$. Then $p = (\bigsqcup Z) \sqcap p = \bigsqcup\{ z \sqcap p \mid z \in Z \}$ by the distributive law (2). Write $Z' = \{ z \sqcap p \mid z \in Z \}$, so $p = \bigsqcup Z'$. Then $x' = x \sqcup p = x \sqcup (\bigsqcup Z') = \bigsqcup\{ x \sqcup z' \mid z' \in Z' \}$. Clearly $x \sqsubseteq x \sqcup z' \sqsubseteq x'$ for all $z \in Z'$. As $x \prec x'$ we must have $x' = x \sqcup z'$ for some $z' \in Z'$; otherwise $x = x \sqcup z'$ for all $z' \in Z'$ giving the contradiction $x = \bigsqcup\{ x \sqcup z' \mid z' \in Z' \} = x'$. But then $p \sqsubseteq z'$ from the definition of p. However $z' = z \sqcap p$ for some $z \in Z$. Therefore $p \sqsubseteq z$ for some $z \in Z$. Thus p is a complete prime of $\mathbf{L}$.

That $\mathbf{L}$ is prime algebraic follows provided for $z \in L$, we have $z = \bigsqcup\{ pr[x, x'] \mid x \prec x' \sqsubseteq z \}$. Let $z \in L$. Write $w = \bigsqcup\{ pr[x, x'] \mid x \prec x' \sqsubseteq z \}$. Clearly $w \sqsubseteq z$. Suppose $w \neq z$. Then, by the lemma, $w \sqsubseteq x \prec x' \sqsubseteq z$ for some $x, x' \in L$. Write $p = pr[x, x']$. Then $p \sqsubseteq w$ making $x \sqcup p = x$, a contradiction as $x \sqcup p = x'$. Thus each element of $\mathbf{L}$ is the least upper bound of the complete primes below it, as required.

Thus we have established the required equivalence between prime algebraic complete lattices and algebraic lattices satisfying (2) and (3). ∎

8 Corollary. *Let $\mathbf{L}$ be a complete lattice. The following are equivalent:*

- *(i)* *$\mathbf{L}$ is isomorphic to $(\mathcal{L}(\mathbf{P}), \subseteq)$ for some partial order $\mathbf{P}$,*
- *(ii)* *$\mathbf{L}$ is prime algebraic,*
- *(iii)* *$\mathbf{L}$ is algebraic and completely distributive,*
- *(iv)* *$\mathbf{L}$ is algebraic and satisfies the distributive laws (2) and (3).*

Proof. Combining previous results. ∎

In the special case when the algebraic lattice satisfies a finiteness restriction we can obtain a similar representation of algebraic lattices mentioning just the finite distributive law (4). The finiteness restriction says every isolated element dominates only a finite number of elements. The corresponding axiom has been called *axiom F*, sometimes *axiom I*, in [KP, BC, W]. (This restriction arises naturally for computations. When a partial order models the events and causal dependency relation of a computation it is generally true that an event is causally dependent on only a finite set of events. The associated left–closed subsets then satisfy the finiteness restriction.)

9 Definition. An algebraic lattice $\mathbf{L} = (L, \sqsubseteq)$ is said to satisfy *axiom F* when $\{ y \in L \mid y \sqsubseteq x \}$ is finite for all isolated elements $x \in L$.

10 Theorem. *Let* $\mathbf{L}$ *be an an algebraic lattice which satifies axiom F. Then* $\mathbf{L}$ *is prime algebraic iff* $\mathbf{L}$ *satisfies the finite distributive law (4).*

Proof. The "only if" part follows from theorem 7. The converse, "if" part, follows from theorem 7 provided we can show that, in the presence of axiom F, the finite distributive law (4) implies the infinite distributive laws (2) and (3).

Let $\mathbf{L}$ be an an algebraic lattice which satifies axiom F and the finite distributive law (4).

We show $\mathbf{L}$ satisfies the infinite distributive law (2). Let $X \subseteq L$ and $y \in L$. Clearly $\bigsqcup\{ x \sqcap y \mid x \in X \} \sqsubseteq (\bigsqcup X) \sqcap y$. To show the converse inequality, suppose b is isolated and $b \sqsubseteq (\bigsqcup X) \sqcap y$. Then as $b \sqsubseteq \bigsqcup X$ and b is isolated, for some finite $X' \subseteq^{\text{fin}} X$ we have $b \sqsubseteq \bigsqcup X'$. Thus

$$\begin{aligned} b \sqsubseteq (\textstyle\bigsqcup X) \sqcap y &\Rightarrow b \sqsubseteq (\textstyle\bigsqcup X') \sqcap y \\ &\Rightarrow b \sqsubseteq \textstyle\bigsqcup\{ x \sqcap y \mid x \in X' \} \text{ (by the finite distributive law (4))} \\ &\Rightarrow b \sqsubseteq \textstyle\bigsqcup\{ x \sqcap y \mid x \in X \}. \end{aligned}$$

Therefore, because $\mathbf{L}$ is algebraic, we have the converse inequality. Combining the inequalities we obtain (2), $\bigsqcup\{ x \sqcap y \mid x \in X \} = (\bigsqcup X) \sqcap y$.

Now we show $\mathbf{L}$ satisfies the infinite distributive law (3). Let $X \subseteq L$ and $y \in L$. Clearly $(\prod X) \sqcup y \sqsubseteq \prod\{ x \sqcup y \mid x \in X \}$. We require the converse inequality. Suppose b is isolated and $b \sqsubseteq \prod\{ x \sqcup y \mid x \in X \}$. Then $b = (\prod\{ x \sqcup y \mid x \in X \}) \sqcap b = \prod\{ (x \sqcup y) \sqcap b \mid x \in X \} = \prod\{ (x \sqcap b) \sqcup (y \sqcap b) \mid x \in X \}$. Now b dominates only a finite number of elements. Thus there is some finite subset $X' \subseteq^{\text{fin}} X$ for which $\{ x \sqcap b \mid x \in X' \} = \{ x \sqcap b \mid x \in X \}$. So in addition, $b = \prod\{ (x \sqcap b) \sqcup (y \sqcap b) \mid x \in X \} = \prod\{ (x \sqcap b) \sqcup (y \sqcap b) \mid x \in X' \}$. Now by the finite distributive law (4), $b = (\prod\{ x \sqcap b \mid x \in X' \}) \sqcup (y \sqcap b) = (\prod\{ x \sqcap b \mid x \in X \}) \sqcup (y \sqcap b) = (\prod X \sqcap b) \sqcup (y \sqcap b) \sqsubseteq (\prod X) \sqcup y$. By algebraicity we obtain $\prod\{ x \sqcup y \mid x \in X \} \sqsubseteq (\prod X) \sqcup y$. Combining the inequalities we obtain (3). ∎

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